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On Bernstein Algebras with Low-dimension Subspaces U^2

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Abstract

The aim of this paper is to study some relations between dimensions of the subspaces UZ, Z^2, U^2 and U^3 of a finite-dimensional Bernstein algebra $A=Ke\dotplus U\dotplus Z$. The main results are concerned in the dependency of $\dim U^3$ on $\dim U^2$ in the case of $\dim U^2 \leqq 4$.

§ 0. Introduction

A nonassociative, commutative algebra A over a field K is called a Bernstein algebra if there exists a nonzero algebra homomorphism $\omega : A \to K$ that satisfies

$$(x^2)^2=\omega(x)^2x^2$$

for every $x \in A$. We suppose that K is an infinite field of characteristic different from 2 and that the dimension of a vector space A over K is finite.

It is known that a Bernstein algebra has always idempotents. If e is an idempotent of A, then A has a Peirce decomposition $A=Ke\dotplus U\dotplus Z$, where $Ker(\omega)=U\dotplus Z$, $U=\{x\in A \mid ex=\frac{1}{2}x\}$, $Z=\{x\in A \mid ex=0\}$.

It is well known that the subspaces U and Z satisfy the following:

(a) $U^2\subset Z$, (b) $UZ+Z^2\subset U$, (c) $UZ^2=\langle 0\rangle$.

It is also well known that the following identities holds for u, u_1, u_2, u_3, $u_4\in U$ and for z, z_1, $z_2\in Z$:

(d) $u^3=0$, $u_1^2u_2=-2u_1(u_1u_2)$,
$u_1(u_2u_3)=-(u_1u_2)u_3-(u_1u_3)u_2$;

(e) $(u^2)^2=0$, $u_1^2(u_2u_3)=-2(u_1u_2)(u_1u_3)$,
$u_1^2u_2^2=-2(u_1u_2)^2$, $u_1^2(u_1u_2)=0$,
$(u_1u_2)(u_3u_4)+(u_1u_3)(u_2u_4)+(u_1u_4)(u_2u_3)=0$;

(f) $u(uz)=0$, $u_1(u_2z)+u_2(u_1z)=0$;

(g) $(uz)^2=0$, $(u_1z)(u_2z)=0$,
$(uz_1)(uz_2)=0$,
$(u_1z_1)(u_2z_2)=-(u_1z_2)(u_2z_1)$.

On the other hand, it is known that the set of idempotents of A is given by $\{e+u+u^2 \mid u\in U\}$ and that if $A=K\bar{e}\dotplus\bar{U}\dotplus\bar{Z}$ is a Peirce decomposition of A with respect to another idempotent $\bar{e}:=e+\bar{u}+\bar{u}^2$, then

(h) $\bar{U}=\{u+2u\bar{u} \mid u\in U\}$ and
$\bar{Z}=\{z-2(\bar{u}+\bar{u}^2)z \mid z\in Z\}$

Moreover, it is known that, $\dim U$ (and so also $\dim Z$) is an invariant of A, that is, it does not depend on the choice of the particular idempotent. Furthermore $\dim U^2$ and $\dim(UZ+Z^2)$ are also invariants of A.[1)] These invariants play a fundamental role in the problem of classifying all finite-dimensional Bernstein algebras, that is yet to be solved. In connection with that we are interested in possible combinations of the values of $\dim U^2$, $\dim U^3$, $\dim UZ$, and $\dim Z^2$.

In the following the subspace spanned by $x_1, x_2, \cdots, x_l\in A$ over K is denoted by $\langle x_1,\cdots, x_l\rangle_K$, or simply $\langle x_1,\cdots,x_l\rangle$ if there exist no apprehensions of misinterpretations.

§ 1. Sufficient conditions for $\dim U^2 \leqq 1$

We state some sufficient conditions for $\dim U^2$ to be equal to or less than 1.

Proposition 1. If $\dim Z^2 = \dim U$, then $U^2 = \langle 0 \rangle$.

proof. Since $U = Z^2$ by assumption and (a), it is clear that $U^2 = \langle 0 \rangle$ from (c). □

Proposition 2. If $\dim Z^2 = \dim U - 1$, then $\dim U^2 \leqq 1$.

proof. Assume that $\dim Z^2 = \dim U - 1$. Then there exists $u \neq 0 \in U$ such that $U = Z^2 \dot{+} Ku$. Hence $U^2 = UZ^2 + uU \subset u(Z^2 + Ku) \subset Ku^2$ by (c). $\therefore$ $\dim U^2 \leqq \dim Ku^2 \leqq 1$. □

Proposition 3. If there exists a nonzero element z in Z such that $U = Uz$, then $U^2 = \langle 0 \rangle$.

proof. By assumption there exist a basis $\{u_i \mid 1 \leqq i \leqq p\}$ of U and an element $z \neq 0$ of Z such that $\{u_i z \mid 1 \leqq i \leqq p\}$ is a basis of U. Then, since $u_i = \sum_{j=1}^{p} \alpha_{ij} u_j z$ with $\alpha_{ij} \in K$ for each i, $1 \leqq i \leqq p$, we have $u_i u_k = \Sigma \alpha_{ij} \alpha_{kl} (u_j z)(u_l z) = 0$ from (g). □

Proposition 4. If $\dim U = \dim UZ = 1$, then $U^2 = \langle 0 \rangle$.

proof. Let $z_1, \cdots, z_q$ be a basis of Z and $U = Ku$, $u \neq 0$. Since $UZ = U$, there exist $\alpha_1, \cdots, \alpha_q$ in K suchthat $uz_i = \alpha_i u (1 \leqq i \leqq q)$, where at least one element, e.g. α_1, is not 0. Then $u^2 = \alpha_1^{-2} (uz_1)^2 = 0$ by (g). □

§ 2. The case $U^2 = \langle 0 \rangle$ with $UZ = U$

Proposition 5. If $\dim UZ = \dim U$, $U^2 = \langle 0 \rangle$, and $\dim Z = 1$, then it is reduced to the case $Z^2 = \langle 0 \rangle$.

proof. The condition that $\dim UZ = \dim U$ is equivalent to $UZ = U$ by (b). Thus we show that, if $Z^2 \neq \langle 0 \rangle$, one can choose proper idempotent $\bar{e}$ so that A has a Peirce decomposition $A = K\bar{e} \dot{+} \bar{U} \dot{+} \bar{Z}$ satisfying that $\bar{U}\bar{Z} = \bar{U}$, $\bar{U}^2 = \langle 0 \rangle$, and $\bar{Z}^2 = \langle 0 \rangle$.

Now choose one nonzero element z_1 of Z and put $u_1 = z_1^2$. If $p = \dim U$, then $Z^2 = Kz_1^2$ and there exists a basis $\{u_1, u_2, \cdots, u_p\}$ of U with $u_1 := z_1^2$. Since $UZ = U$, $u_1 z_1, \cdots, u_p z_1$ are linearly independent. Hence, if we write $u_i z_1 = \sum_j \alpha_{ij} u_j$ $(i = 1, \cdots, p)$, then the determinant $\Delta := \det[\alpha_{ij}]$ is not 0. Then, the linear equation $\sum_i \lambda_i (u_i z_1) z_1 = 0$ with $\lambda_i \in K$ implies that $\sum_{i=1}^{p} \lambda_i [\sum_{j=1}^{p} \alpha_{ij} u_j z_1] = \sum_{j=1}^{p} [\sum_{i=1}^{p} \lambda_i \alpha_{ij}] u_j z_1 = 0$, therefore, $\sum_{i=1}^{p} \lambda_i \alpha_{ij} = 0$ for each j and the determinant of this system of linear equations is identical to Δ. Because $\Delta \neq 0$, we have $\lambda_i = 0$ for all i, which means that $(u_1 z_1) z_1, \cdots, (u_p z_1) z_1$ are linearly independent and there exist uniquely $\beta_i (i = 1, \cdots, p)$ in K such that $u_1 = \sum_i \beta_i (u_i z_1) z_1$. Now, if we define $\bar{u} := \frac{1}{4} \sum_{i=1}^{n} \beta_i u_i$ and $\bar{e} := e + \bar{u}$, then, from (h), we get $\bar{U} = \{u + 2u\bar{u} \mid u \in U\} = U$ and $\bar{Z} = \{z - 2\bar{u}z \mid z \in Z\} = K(z_1 - 2\bar{u}z_1)$ with $(z_1 - 2\bar{u}z_1)^2 = 0$ by $U^2 = \langle 0 \rangle$ and (g). □

§ 3. Some consequences of $\dim U^2 = 1$

Theorem 1. If $\dim U^2 = 1$, then $U^3 = \langle 0 \rangle$ and $(U^2)^2 = \langle 0 \rangle$.

proof. Let $\{u_i \mid i = 1, \cdots, p\}$ be a basis of U. Then, by assumption, there exists at least one nonzero element in the set $\{u_i u_j \mid 1 \leqq i \leqq j \leqq p\}$. Let $z_1 := u_{i0} u_{j0} \neq 0$ and put $u_i u_j = \alpha_{ij} z_1$ with $\alpha_{ij} \in K$ for each pair $i, j (1 \leqq i \leqq j \leqq p)$. Then there occur two possible cases: $i_0 = j_0$ or $i_0 < j_0$. We prove the assertion in each case.

i) If $i_0 = j_0$, then we can assume that $i_0 = j_0 = 1$, i.e., $z_1 = u_1^2$ without loss of generality. From (d) we have that $z_1 u_i = -2\alpha_{1i} u_1^3 = 0$ for all j, which means that $U^3 = \langle 0 \rangle$. On the other hand, since $U^2 = Kz_1$ and $z_1^2 = 0$ from (d), we get also $(U^2)^2 = \langle 0 \rangle$. □

ii) If $i_0 < j_0$, then we can put $(i_0, j_0) = (1, 2)$, i.e., $z_1 = u_1 u_2$ without loss of generality. Then, by assumption and (d), it holds that

(1) $(u_1^2)^2 = \alpha_{11}^2 z_1^2 = 0$, $(u_2^2)^2 = \alpha_{22}^2 z_1^2 = 0$;

(2) $z_1 u_1 = -\frac{1}{2} u_1^2 u_2 = -\frac{1}{2} \alpha_{11} z_1 u_2 = \frac{1}{4} \alpha_{11} u_1 u_2^2$

$$= \frac{1}{4} \alpha_{11} \alpha_{22} z_1 u_1;$$

and in like manner

(3) $z_1 u_2 = \frac{1}{4} \alpha_{11} \alpha_{22} z_1 u_2$;

(4) $z_1 u_i = \frac{1}{2} \alpha_{1i} \alpha_{22} z_1 u_1 + \frac{1}{2} \alpha_{2i} \alpha_{11} z_1 u_2$ for all $i (3 \leqq i \leqq p)$.

The equation (1) implies the following

(5) $z_1^2=0$ or $\alpha_{11}=\alpha_{22}=0$.

Thus, if $\alpha_{11}\alpha_{22}\neq 4$, then $U^3=\langle 0\rangle$ by virtue of (2), (3) and (4), and moreover $z_1^2=-\frac{1}{2}u_1^2u_2^2=-\frac{1}{2}\alpha_{11}\alpha_{22}z_1^2$ by (5). On the contrary, if $\alpha_{11}\alpha_{22}=4$, then this case belongs to the case $i_0=j_0=1$, since $u_1^2=\alpha_{11}z_1\neq 0$. □

Corollary 1. If $\dim U^2=\dim Z=1$, then $UZ+Z^2=\langle 0\rangle$.

proof. The claim follows from Theorem 1 since $U^2=Z$. □

Theorem 2. If $\dim U^2=1$ and $\dim Z=2$, then $\dim UZ<\dim U$.

proof. Let $\{u_1, \cdots, u_p\}$ be a basis of U and choose a basis $\{z_1, z_2\}$ of Z such that $U^2=Kz_1$. Then we get $Uz_1=0$ as a corollary of Theorem 1. Therefore $UZ=<u_1z_2, u_2z_2, \cdots, u_pz_2>$. Put $u_iu_j=\alpha_{ij}z_1$ and $u_iz_2=\Sigma\gamma_{ik}u_k$ with α_{ij}, $\gamma_{ik}\in K$ for every i, $j(1\leqq i\leqq j\leqq p)$. Then, since $u_i(u_jz_2)+u_j(u_iz_2)=0$ and $(u_iz_2)(u_jz_2)=0$ from (f) and (g), respectively, the following equations hold for each pair (i, j):

(1) $\sum_k\alpha_{ik}\gamma_{jk}+\sum_k\alpha_{jk}\gamma_{ik}=0$,

(2) $\sum_{k,l}\alpha_{kl}\gamma_{ik}\gamma_{jk}=0$.

Define $\gamma_k:=\sum_i\gamma_{ik}$ and $\alpha_{\cdot k}:=\sum_j\alpha_{jk}$ for each k. Then, from (1) $0=\sum_j[\sum_k\alpha_{ik}\gamma_{jk}]+\sum_j[\sum_k\alpha_{jk}\gamma_{ik}]=\sum_k[\sum_j\gamma_{jk}]\alpha_{ik}+\sum_k[\sum_j\alpha_{jk}]\gamma_{ik}$. Therefore

(3) $\sum_k\alpha_{ik}\gamma_k+\sum_k\alpha_{\cdot k}\gamma_{ik}=0$ for $i=1, \cdots p$.

There are two possible cases.

If $\gamma_k=0$ for all k, then, putting $u_0:=\sum_i u_i$, we have that $u_0z_2=\sum_i u_iz_2=\sum_i[\sum_k\gamma_{ik}u_k]=\sum_k[\sum_i\gamma_{ik}]u_k=\sum_k\gamma_ku_k=0$. Therefore, by adopting $\{u_0, u_2, \cdots, u_p\}$ as a basis of U, we get that $Uz_2=\sum_{i>1}Ku_iz_2$. So $\dim Uz_2<\dim U$.

If $\gamma_{ko}\neq 0$ for some k_o, then there exist two situations:

i) If $\sum_k\alpha_{ik}\gamma_k\neq 0$ for some i, then $\xi_1, \cdots, \xi_p$ defined by $\xi_i:=\sum_k\alpha_{ik}\gamma_k(i=1, \cdots, p)$ satisfy the simultaneous equations $\sum_k\gamma_{jk}\xi_k=0(j=1, \cdots, p)$, which is shown from (2), and $\xi_i\neq 0$ for some i by assumption. Therefore $\det[\gamma_{jk}]_{j\cdot k}=0$, which means that $u_1z_2, \cdots, u_pz_2$ are linearly dependent and $\dim Uz_2<p$.

ii) If $\sum_k\alpha_{ik}\gamma_k=0$ for all i, then euations (3) is reduced to

(4) $\sum_k\alpha_{\cdot k}\gamma_{ik}=0$ for $i=1, \cdots, p$.

Since $u_i\sum_k u_k=\sum_k\alpha_{ik}z_1=\alpha_{\cdot i}z_1$ for all i, the claim that $\alpha_{\cdot k}=0$ for all k is contrary to the assumption that $\dim U^2=1$ and we can conclude that $\alpha_{\cdot k}\neq 0$ for some k. Then the simultaneous equations $\Sigma\gamma_k{}^i\xi_k=0(i=1, \cdots, p)$ have non-trivial solutions $\xi_k=\alpha_{\cdot k}(k=1, \cdots, p)$, which means that $\dim Uz_2<p$. □

§ 4. The case $\dim U^2=2$, 3, or 4

First of all we state two lemmas which will be used in the proofs of the theorems following below. The first lemma is elementary.

Lemma 1. Let $B=\{a_1, a_2, \cdots, a_k\}(k>0)$ be a basis of a k-dimensional vector space V and $b=\Sigma\lambda_ia_i$ a nonzero vector with some $\lambda_i\neq 0$. Then, also the set $\{a_1, a_2, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_k\}$ obtained from B by replacing a_i with b is a basis of V.

Lemma 2. Let $\{u_i|i=1, \cdots, p\}$ be a basis of U and $\{z_r=u_{i_rj_r}|r=1, \cdots, k\}$ a basis of U^2, where we suppose that $k=\dim U^2$, $1\leqq i_1\leqq i_2\leqq\cdots\leqq i_k\leqq p$, $1\leqq j_1\leqq j_2\leqq\cdots\leqq j_k\leqq p$, and $i_r\leqq j_r$ for $r=1, 2, \cdots, k$. If X is a subspace of U and $z_ru_t\in X$ for each $r(1\leqq r\leqq k)$ and each $t\in\{i_1, j_1, i_2, j_2, \cdots, i_k, j_k\}$, then $z_ru_t\in X$ for each $r(1\leqq r\leqq k)$ and each $t(1\leqq t\leqq p)$.

proof. We shall put $z_s=u_iu_j(i=i_s, j=j_s)$ for any $s(1\leqq s\leqq k)$ and $u_iu_t=\Sigma\alpha_{it}{}^{(r)}z_r$, $u_ju_t=\Sigma\alpha_{jt}{}^{(r)}z_r$ for each $t(1\leqq t\leqq p)$ with $t\in\{i_1, j_1, i_2, j_2, \cdots, i_k, j_k\}$. Then, by (d), $z_su_t=-(u_iu_t)u_j-(u_ju_t)u_i=-\Sigma\alpha_{it}{}^{(r)}z_ru_j-\Sigma\alpha_{jt}{}^{(r)}z_ru_i$, where z_ru_i, $z_ru_j\in X$ by assumption. Therefore $z_3u_t\in X$. □

Theorem 3. If $\dim U^2=2$, then $U^3=\langle 0\rangle$ or else $(U^2)^2=\langle 0\rangle$ and, in either case, $\dim U^3\leqq 1$.

proof. Let $u_1, \cdots, u_p(p\geqq 2)$ be a basis of U. Then, by assumption, we can choose two products u_iu_j and u_ku_l with $i \leqq j$, $k \leqq l$, $i \leqq k$, $(i, j) \neq (k, l)$, as a basis of U^2. On the other hand we can easily see that each possible combination of u_i, u_j; u_k, u_l, denoted simply $[ij, kl]$, belongs to one of the following five types, by changing the number i of u_i, if necessary:

1: $[11, 22]$ (i.e. u_1^2, u_2^2)
2: $[11, 12]$ (i.e. u_1^2, u_1u_2)
3: $[11, 23]$ (i.e. u_1^2, u_2u_3)
4: $[12, 13]$ (i.e. u_1u_2, u_1u_3)
5: $[12, 34]$ (i.e. u_1u_2, u_3u_4)

Once a basis z_1, z_2 of U^2 is chosen, each remaining product $u_iu_j \in U^2$ will be written $u_iu_j = \alpha_{ij}z_1 + \beta_{ij}z_2$ with α_{ij}, $\beta_{ij} \in K$. We shall establish the assertion of the theorem for each type, separately.

Type 1: [11, 22]

We put $z_1 = u_1^2$ and $z_2 = u_2^2$. Then by (d)

(1) $z_1u_1 = z_2u_2 = 0$, $z_1u_2 = 4\alpha_{12}\beta_{12}z_1u_2$,

and by (e)

(2) $z_1^2 = z_2^2 = 0$, therefore

$$z_1z_2 = -2(u_1u_2)^2 = -4\alpha_{12}\beta_{12}z_1z_2.$$

If $4\alpha_{12}\beta_{12} = -1$, then we can conclude from (1) that $z_iu_j = 0$ for $i, j = 1, 2$, which implies by virtue of Lemma 2 that $z_1u_i = z_2u_i = 0$ for all $i(1 \leqq i \leqq p)$. Consequently we have $U^3 = 0$. On the contrary, if $4\alpha_{12}\beta_{12} \neq 0$, we obtain that $U^3 \subset Kz_1u_2$ and $\dim U^3 \leqq 1$ from (1). Also $(U^2)^2 = 0$ from (2). □

Type 2: [11, 12]

We put $z_1 = u_1^2$ and $z_2 = u_1u_2$. If $\beta_{22} \neq 0$, then this type is reduced to Type 1, as is seen easily by Lemma 1. Therefore we can suppose $\beta_{22} = 0$, that is, $u_2^2 = \alpha_{22}z_1$. Define X by $X := Kz_1u_2$. Then by (d) and by assumption

(1) $z_1u_1 = 0$, $z_2u_1 = -\frac{1}{2}z_1u_2 \in X$, so

$$z_2u_2 = -\frac{1}{2}\alpha_{22}z_1u_1 = 0,$$

therefore by Lemma 2

(2) $z_1u_i \in X$, $z_2u_i \in X$ for all $i(1 \leqq i \leqq p)$.

Consequently we have $U^3 \subset X$ and so forth. Moreover, (e) and $(u_1u_2)^2 = -\frac{1}{2}u_1^2u_2^2 = -\frac{1}{2}\alpha_{22}z_1^2 = 0$ imply that $(U^2)^2 = \langle 0 \rangle$. □

Type 3: [11, 23]

We put $z_1 = u_1^2$, and $z_2 = u_2u_3$. If there exists at least one nonzero element in $\{\beta_{22}, \beta_{33}\}$ or $\{\alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}\}$, then this type is reduced to Type 1 or Type 2, respectively, as is shown by Lemma 1. Therefore we can suppose that $\alpha_{22} = \beta_{22} = \alpha_{33} = \beta_{33} = \beta_{12} = \beta_{13} = 0$, i.e., $u_2^2 = u_3^2 = 0$, $u_1u_2 = \alpha_{12}u_1^2$ and $u_1u_3 = \alpha_{13}u_1^2$. Then by assumption and (d)

(1) $z_1u_i = z_2u_i = 0$ for $i = 1, 2, 3$.

Consequently by Lemma 2

(2) $z_1u_i = z_2u_i = 0$ for all $i(1 \leqq i \leqq p)$,

which means that $U^3 = \langle 0 \rangle$. □

Type 4: [12, 13]

We put $z_1 = u_1u_2$ and $z_2 = u_1u_3$. If there exists at least one nonzero element in $\{\alpha_{11}, \beta_{11}, \beta_{22}, \alpha_{33}\}$ or $\{\alpha_{22}, \beta_{33}\}$, then this type is reduced to Type 2 or Type 3, respectively, as is shown by Lemma 1. Hence we can suppose that $\alpha_{ii} = \beta_{ii} = 0$, i.e., $u_i^2 = 0$ for $i = 1, 2, 3$ and $u_2u_3 = \alpha_{23}z_1 + \beta_{23}z_2$. Now define the subspace X of U by $X := Kz_1u_3$, Then by assumption and (d)

(1) $z_1u_1 = z_1u_2 = z_2u_1 = z_2u_3 = 0$, $z_2u_2 = -z_1u_3$.

Therefore by Lemma 2

(2) $z_1u_i \in X$, $z_2u_i \in X$ for $i(1 \leqq i \leqq p)$,

which means that $U^3 \subset X$ and so forth. On the other hand, by assumption and (e), $z_1^2 = z_2^2 = z_1z_2 = 0$. This implies that $(U^2)^2 = \langle 0 \rangle$. □

Type 5: [12, 34]

We put $z_1 = u_1u_2$ and $z_2 = u_3u_4$. If there exists at least one nonzero element in $\{\beta_{11}, \beta_{22}, \alpha_{33}, \alpha_{44}\}$, $\{\alpha_{11}, \alpha_{22}, \beta_{33}, \beta_{44}\}$ or $\{\alpha_{13}, \alpha_{23}, \beta_{13}, \beta_{23}, \alpha_{14}, \alpha_{24}, \beta_{14}, \beta_{24}\}$, then this type is reduced to Type 2, 3 or 4, respectively, as is shown by Lemma 1. Therefore we can assume that $u_iu_j = 0$ for all $(i, j) \neq (1, 2), (3, 4) (1 \leqq i, j \leqq 4)$. Then by assumption and (d)

(1) $z_1u_1 = z_1u_2 = z_2u_3 = z_2u_4 = 0$,
$z_1u_3 = z_1u_4 = z_2u_1 = z_2u_2 = 0$.

Then by Lemma 2

(2) $z_1u_i = z_2u_i = 0$ for all $i(1 \leqq i \leqq p)$.

Therefore $U^3 = \langle 0 \rangle$ and so forth. □

Corollary 2. If $\dim U^2 = 2$ and $U^2 = Z$, then $UZ = \langle 0 \rangle$ or else $Z^2 = \langle 0 \rangle$, and, in either case, $\dim UZ \leqq 1$.

Theorem 4. If $\dim U^2=3$, then $\dim U^3 \leqq 2$ or else $(U^2)^2=\langle 0\rangle$ and, in either case, $\dim U^3 \leqq 3$.

proof. We shall prove the theorem in the same method as one for Theorem 3 that is composed of classifying the types of base elements of U^2 and computing separately according to the types, reducing to the established types before by virtue of Lemma 1 and Lemma 2. However, in order to avoid redundance, we shall describe in the following only results of verification omitting the detail of computing.

Let $u_1, \cdots, u_p$ be a basis of U. Then, by assumption, we can choose three products u_iu_j, u_ku_l and u_mu_n with $i \leqq j$, $k \leqq l$, $m \leqq n$, $i \leqq k \leqq m$, $(i, j) \neq (k, l) \neq (m, n)$, as a basis of U^2. Then each possible combination in three products, denoted by $[ij, kl, mn]$ for short, belongs to one of the fourteen types listed below by changing the number i of u_i, if necessary:

1: [11, 12, 22]	2: [12, 22, 23]
3: [11, 12, 23]	4: [11, 22, 33]
5: [11, 22, 23]	6: [12, 13, 23]
7: [11, 22, 34]	8: [11, 12, 34]
9: [11, 23, 24]	10: [12, 13, 14]
11: [12, 13, 24]	12: [11, 23, 45]
13: [12, 13, 45]	14: [12, 34, 56]

Once a basis z_1, z_2, z_3 of U^2 are chosen, every remaining product $u_iu_j \in U^2$ will be written $u_iu_j=\alpha_{ij}z_1+\beta_{ij}z_2+\gamma_{ij}z_3$ with $\alpha_{ij}, \beta_{ij}, \gamma_{ij} \in K$.

Type 1: [11, 12, 22]

We put $z_1=u_1^2$, $z_2=u_1u_2$, and $z_3=u_2^2$. Then it is shown that $U^3 \subset \langle z_1u_2, z_3u_1\rangle$ and $\dim U^3 \leqq 2$. □

Type 2: [12, 22, 23]

We put $z_1=u_1^2$, $z_2=u_2^2$, and $z_3=u_2u_3$. Omitting the case that is reduced to Type 1, we can assume that $\alpha_{33}=\gamma_{12}=0$. Then it is shown that $U^3 \subset \langle z_1u_2, z_1u_3, z_2u_3\rangle$ and that z_1u_2, z_1u_3, z_2u_3 are linearly dependent, so $\dim U^3 \leqq 2$.

Type 3: [11, 12, 23]

We put $z_1=u_1^2$, $z_2=u_1u_2$, and $z_3=u_2u_3$. Omitting the case that is reduced to Type 1 or Type 2, we can assume that $\beta_{22}=\beta_{33}=\gamma_{22}=\gamma_{33}=0$. Then it is shown that $U^3 \subset \langle z_1u_2, z_3u_1\rangle$ and so forth. □

Type 4: [11, 22, 33]

We put $z_1=u_1^2$, $z_2=u_2^2$, and $z_3=u_3^2$. Omitting the case that is reduced to Type 1 or Type 2, we can assume that $\alpha_{ij}=\beta_{ij}=\gamma_{ij}=0$ for all $i, j (1 \leqq i \leqq j \leqq 3)$. Then it is shown that $U^3=\langle 0\rangle$ and that, together with the preceding results, $\dim U^3 \leqq 2$. □

Type 5: [12, 22, 23]

We put $z_1=u_1u_2$, $z_2=u_2^2$, and $z_3=u_3^2$. Omitting the case that is reduced to Type 1, 2, or 3, we can assume that $\alpha_{11}=\gamma_{11}=\gamma_{33}=\beta_{11}=\beta_{33}=\alpha_{13}=\gamma_{13}=0$. Then it is shown that $U^3 \subset \langle z_1u_3, z_2u_1, z_2u_3\rangle$ and that $(U^2)^2=\langle 0\rangle$. □

Type 6: [12, 13, 23]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, and $z_3=u_2u_3$. Omitting the case that is reduced to Type 2 or Type 5, we can assume that $\alpha_{ii}=\beta_{ii}=\gamma_{ii}=0$ for $i=1, 2, 3$. Then it is shown that $U^3 \subset \langle z_1u_3, z_3u_1\rangle$ and so forth. □

Type 7: [11, 22, 34]

We put $z_1=u_1^2$, $z_2=u_2^2$, and $z_3=u_3u_4$. Omitting the case that is reduced to Type 1, 2, 3, or 4, we can assume that $\alpha_{33}=\beta_{33}=\gamma_{33}=\alpha_{44}=\beta_{44}=\gamma_{44}=\gamma_{12}=\beta_{13}=\gamma_{13}=\beta_{14}=\gamma_{14}=\alpha_{23}=\gamma_{23}=\alpha_{24}=\gamma_{24}=0$. Then it is shown that $U^3 \subset Kz_1u_2$ and so forth. □

Type 8: [11, 12, 34]

We put $z_1=u_1^2$, $z_2=u_1u_2$, and $z_3=u_3u_4$. Omitting the case that is reduced to Type 1, 2, 3, 5, or 7, we can assume that $\beta_{22}=\gamma_{22}=\beta_{33}=\gamma_{33}=\beta_{44}=\gamma_{44}=\gamma_{12}=\beta_{13}=\gamma_{13}=\beta_{14}=\gamma_{14}=\gamma_{23}=\gamma_{24}=0$. Then it is shown that $U^3 \subset Kz_1u_2$ and so forth. □

Type 9: [11, 23, 24]

We put $z_1=u_1^2$, $z_2=u_2u_3$, and $z_3=u_2u_4$. Omitting the case that is reduced to Type 2, 3, 5, 7, or 8, we can assume that $\alpha_{ii}=\beta_{ii}=\gamma_{ii}=\beta_{1i}=\gamma_{1i}=0$ for $j=2, 3, 4$. Then it is shown that $U^3 \subset Kz_2u_4$ and so forth. □

Type 10: [12, 13, 14]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, and $z_3=u_1u_4$. Omitting the case that is reduced to Type 3, 5, 6, or 9, we can assume that $\alpha_{ii}=\beta_{ii}=\gamma_{ii}=0$ for $i=1, 2, 3, 4$ and $\alpha_{34}=\beta_{24}=\gamma_{23}=0$. Then it is shown that $U^3 \subset \langle z_1u_3, z_1u_4, z_2u_4 \rangle$ and that $(U^2)^2 = \langle 0 \rangle$. □

Type 11: [12, 13, 24]

We put $z_1 = u_1u_2$, $z_2 = u_1u_3$, and $z_3 = u_2u_4$. Omitting the case that is reduced to Type 3, 5, 6, 8, 9, or 10, we can assume that $\alpha_{ii} = \beta_{ii} = \gamma_{ii} = 0$ for $j = 1, 2, 3, 4$ and $\beta_{ij} = \gamma_{ij} = 0$ for $(i, j) = (1, 4), (2, 3)$. Then it is shown that $U^3 \subset \langle z_1u_3, z_1u_4 \rangle$ and so forth. □

Type 12: [11, 23, 45]

We put $z_1 = u_1^2$, $z_2 = u_2u_3$, and $z_3 = u_4u_5$. Omitting the case that is reduced to Type 2, 3, 7, 8, 9, or 11, we can assume that $\alpha_{ii} = \beta_{ii} = \gamma_{ii} = 0$ for $i = 2, 3, 4, 5$ and $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0$ for $i = 2, 3$ and $j = 4, 5$ and $\beta_{1i} = \gamma_{1i} = 0$ for $i = 2, 3, 4, 5$. Then it is shown that $U^3 = \langle 0 \rangle$ and so forth. □

Type 13: [12, 13, 45]

We put $z_1 = u_1u_2$, $z_2 = u_1u_3$, and $z_3 = u_4u_5$. Omitting the case that is reduced to Type 3, 5, 6, 8, 9, 10, 11 or 12, we can assume that $\alpha_{ii} = \beta_{ii} = \gamma_{ii} = 0$ for $i = 1, 2, 3, 4, 5$ and $\alpha_{14} = \alpha_{15} = \alpha_{34} = \alpha_{35} = \beta_{14} = \beta_{15} = \beta_{24} = \beta_{25} = \gamma_{14} = \gamma_{15} = \gamma_{23} = \gamma_{24} = \gamma_{25} = \gamma_{34} = \gamma_{35} = 0$. Then it is shown that $U^3 \subset Kz_1u_3$ and so forth. □

Type 14: [12, 34, 56]

We put $z_1 = u_1u_2$, $z_2 = u_3u_4$, and $z_3 = u_5u_6$. Omitting the case that is reduced to Type 8, 10, 11, 12, or 13, we can assume that $u_iu_j = 0$ for every pair $(i, j) \neq (1, 2), (3, 4), (5, 6)$ $(1 \leqq i, j \leqq 6)$. Then it is shown that $U^3 = \langle 0 \rangle$ and so forth. □

Corollary 3. If $\dim U^2 = 3$ and $U^2 = Z$, then $\dim UZ \leqq 2$ or else $Z^2 = \langle 0 \rangle$, and, in either case, $\dim UZ \leqq 3$.

Theorem 5. If $\dim U^2 = 4$, then $\dim U^3 \leqq 5$ or else $(U^2)^2 = \langle 0 \rangle$ and, in either case, $\dim U^3 \leqq 6$.

proof. We shall state here also only results of verification omitting the detail of computing.

Let $u_1, \cdots, u_p$ be a basis of U. Then, by assumption, we can choose four products $z_1 = u_iu_j$, $z_2 = u_ku_l$, $z_3 = u_mu_n$, and $z_4 = u_3u_t$, where $i \leqq j$, $k \leqq l$, $m \leqq n$, $s \leqq t$, $i \leqq k \leqq m \leqq s$, $(i, j) \neq (k, l) \neq (m, n) \neq (s, t)$, as a basis of U^2. Then each possible combination in the four products, denoted by $[ij, kl, mn, st]$, belongs to one of the following thirty-nine types by changing the number i of u_i, if necessary:

1: [11, 12, 13, 22] 2: [11, 12, 13, 23]
3: [11, 13, 22, 23] 4: [11, 12, 22, 33]
5: [11, 22, 33, 44] 6: [11, 13, 14, 22]
7: [11, 12, 13, 14] 8: [11, 14, 22, 33]
9: [11, 12, 13, 24] 10: [11, 12, 22, 34]
11: [11, 13, 22, 24] 12: [11, 12, 24, 33]
13: [11, 12, 23, 34] 14: [11, 12, 23, 24]
15: [11, 23, 24, 34] 16: [12, 13, 14, 23]
17: [12, 13, 24, 34] 18: [11, 22, 33, 45]
19: [11, 13, 22, 45] 20: [11, 22, 34, 35]
21: [11, 12, 23, 45] 22: [11, 12, 13, 45]
23: [11, 23, 24, 35] 24: [11, 13, 24, 25]
25: [13, 14, 15, 22] 26: [12, 13, 14, 15]
27: [12, 13, 14, 25] 28: [12, 13, 24, 35]
29: [12, 14, 23, 35] 30: [12, 13, 23, 45]
31: [11, 22, 34, 56] 32: [11, 12, 34, 56]
33: [11, 23, 24, 56] 34: [12, 13, 14, 56]
35: [12, 13, 24, 56] 36: [13, 14, 25, 26]
37: [11, 23, 45, 67] 38: [12, 13, 45, 67]
39: [12, 34, 56, 78]

Type 1: [11, 12, 13, 22]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_1u_3$, and $z_4 = u_2^2$. Then it is shown that $U^3 \subset \langle z_1u_2, z_1u_3, z_2u_3, z_4u_1 \rangle$ and so $\dim U^3 \leqq 4$. □

Type 2: [11, 12, 13, 23]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_1u_3$, and $z_4 = u_2u_3$. Omitting the case that is reduced to Type 1, we obtain that $U^3 \subset \langle z_1u_2, z_1u_3, z_3u_3, z_4u_1 \rangle$ and so $\dim U^3 \leqq 4$. □

Type 3: [11, 13, 22, 23]

We put $z_1 = u_1^2$, $z_2 = u_1u_3$, $z_3 = u_2^2$, and $z_4 = u_2u_3$. Omitting the case that is reduced to Type 1 or Type 2, we obtain that $U^3 \subset \langle z_1u_3, z_3u_3, z_4u_1 \rangle$ and that, together with the prededing results, $\dim U^3 \leqq 4$. □

Type 4: [11, 12, 22, 33]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_2^2$, and $z_4 = u_3^2$. Omitting the case that is reduced to Type 1 or Type 3, we obtain that $U^3 \subset \langle z_1u_2, z_3u_1 \rangle$ and that, together with the prededing results, dim $U^3 \leqq 4$. □

Type 5: [11, 22, 33, 44]

We put $z_1 = u_1^2$, $z_2 = u_2^2$, $z_3 = u_3^2$, and $z_4 = u_4^2$. Omitting the case that is reduced to Type 4, we obtain that $U^3 \subset \langle z_1u_2, z_1u_3, z_1u_4, z_2u_3, z_2u_4, z_3u_4 \rangle$ and that $z_1u_2, z_1u_3, z_1u_4, z_2u_3, z_2u_4, z_3u_4$ are linearly dependent, so $\dim U^3 \leqq 5$, or else $(U^2)^2 = \langle 0 \rangle$. □

Type 6: [11, 13, 14, 22]

We put $z_1 = u_1^2$, $z_2 = u_1u_3$, $z_3 = u_1u_4$, and $z_4 = u_2^2$. Omitting the case that is reduced to Type 1, 2, 3, 4, or 5, we obtain that $U^3 \subset \langle z_1u_2, z_1u_3, z_1u_4, z_2u_2, z_3u_2, z_3u_3 \rangle$ and that $z_1u_2, z_1u_3, z_1u_4, z_2u_2, z_3u_2, z_3u_3$ are linearly dependent, so $\dim U^3 \leqq 5$, or else $(U^2)^2 = \langle 0 \rangle$. □

Type 7: [11, 12, 13, 14]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_1u_3$, and $z_4 = u_1u_4$. Omitting the case that is reduced to Type 1, 2, or 6, we obtain that $U^3 \subset \langle z_1u_2, z_1u_3, z_1u_4, z_3u_2, z_4u_2, z_4u_3 \rangle$ and that $(U^2)^2 = \langle 0 \rangle$. □

Type 8: [11, 14, 22, 33]

We put $z_1 = u_1^2$, $z_2 = u_1u_4$, $z_3 = u_2^2$, and $z_4 = u_3^2$. Omitting the case that is reduced to Type 1, 3, 4, 5, or 6, we obtain that $U^3 \subset \langle z_1u_4, z_3u_3 \rangle$ and so forth. □

Type 9: [11, 12, 13, 24]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_1u_3$, and $z_4 = u_2u_4$. Omitting the case that is reduced to Type 1, 2, 3, 4, 6, or 7, we obtain that $U^3 \subset \langle z_1u_2, z_1u_3, z_3u_2, z_4u_1 \rangle$ and so on. □

Type 10: [11, 12, 22, 34]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_2^2$, and $z_4 = u_3u_4$. Omitting the case that is reduced to Type 1, 8, or 9, we obtain that $U^3 \subset \langle z_1u_2, z_3u_1 \rangle$ and so forth. □

Type 11: [11, 13, 22, 24]

We put $z_1 = u_1^2$, $z_2 = u_1u_3$, $z_3 = u_2^2$, and $z_4 = u_2u_4$. Omitting the case that is reduced to Type 1, 3, 4, 6, 9, or 10, we obtain that $U^3 \subset \langle z_1u_3, z_3u_4 \rangle$ and so forth. □

Type 12: [11, 12, 24, 33]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_2u_4$, and $z_4 = u_3^2$. Omitting the case that is reduced to Type 1, 2, 3, 4, 6, 8, 9, 10, or 11, we obtain that $U^3 \subset \langle z_1u_2, z_3u_1 \rangle$ and so forth. □

Type 13: [11, 12, 23, 34]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_2u_3$, and $z_4 = u_3u_4$. Omitting the case that is reduced to Type 1, 2, 3, 6, 8, 9, 10, 11 or 12, we obtain that $U^3 \subset \langle z_1u_2, z_3u_1, z_4u_1, z_4u_2 \rangle$ and so on. □

Type 14: [11, 12, 23, 24]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_2u_3$, and $z_4 = u_2u_4$. Omitting the case that is reduced to Type 1, 2, 3, 4, 6, 7, 8, 9, 12, or 13, we obtain that $U^3 \subset \langle z_1u_2, z_3u_1, z_4u_1, z_4u_3 \rangle$ and so on. □

Type 15: [11, 23, 24, 34]

We put $z_1 = u_1^2$, $z_2 = u_2u_3$, $z_3 = u_2u_4$, and $z_4 = u_3u_4$. Omitting the case that is reduced to Type 2, 6, 12, 13, or 14, we obtain that $U^3 \subset \langle z_3u_3, z_4u_2 \rangle$ and so forth. □

Type 16: [12, 13, 14, 23]

We put $z_1 = u_1u_2$, $z_2 = u_1u_3$, $z_3 = u_1u_4$, and $z_4 = u_2u_3$. Omitting the case that is reduced to Type 2, 7, 9, 13, 14, or 15, we obtain that $U^3 \subset \langle z_2u_2, z_3u_2, z_3u_3, z_4u_1 \rangle$ and so forth. □

Type 17: [12, 13, 24, 34]

We put $z_1 = u_1u_2$, $z_2 = u_1u_3$, $z_3 = u_2u_4$, and $z_4 = u_3u_4$. Omitting the case that is reduced to Type 9, 13, or 16, we obtain that $U^3 \subset \langle z_1u_4, z_2u_2, z_3u_1, z_4u_1, z_4u_2 \rangle$ and so forth. □

Type 18: [11, 22, 33, 45]

We put $z_1 = u_1^2$, $z_2 = u_2^2$, $z_3 = u_3^2$, and $z_4 = u_4u_5$. Omitting the case that is reduced to Type 4, 5, 8, 10, or 12, we obtain that $U^3 \subset \langle z_1u_2, z_1u_3, z_2u_3 \rangle$ and so forth. □

Type 19: [11, 13, 22, 45]

We put $z_1 = u_1^2$, $z_2 = u_1u_3$, $z_3 = u_2^2$, and $z_4 = u_4u_5$. Omitting the case that is reduced to Type 1, 3, 4, 6, 9, 10, 11, 12, 16, or 18, we get that $U^3 \subset \langle z_1u_2, z_1u_3, z_2u_2 \rangle$ and so forth. □

Type 20: [11, 22, 34, 35]

We put $z_1 = u_1^2$, $z_2 = u_2^2$, $z_3 = u_3u_4$, and $z_4 = u_3u_5$. Omitting the case that is reduced to Type 8, 9, 10, 12, 14, 15, 16, 18, or 19, we get that $U^3 \subset \langle z_1u_2, z_3u_5 \rangle$ and so forth. □

Type 21: [11, 12, 23, 45]

We put $z_1 = u_1^2$, $z_2 = u_1u_2$, $z_3 = u_2u_3$, and $z_4 = u_4u_5$. Omitting the case that is reduced to Type

1 , 2 , 3 , 9 , 10 , 11 , 12 , 13, or 19, we obtain that $U^3 \subset \langle z_1u_2, z_3u_1\rangle$ and so forth. □

Type 22: [11, 12, 13, 45]

We put $z_1=u_1^2$, $z_2=u_1u_2$, $z_3=u_1u_3$, and $z_4=u_4u_5$. Omitting the case that is reduced to Type 1 , 2 , 6 , 7 , 9 , 10 , 11 , 13, or 21, we obtain that $U^3 \subset \langle z_1u_2, z_1u_3, z_2u_3\rangle$ and so forth. □

Type 23: [11, 23, 24, 35]

We put $z_1=u_1^2$, $z_2=u_2u_3$, $z_3=u_2u_4$, and $z_4=u_3u_5$. Omitting the case that is reduced to Type 6 , 9 , 12 , 13 , 14 , 15 , 16 , 17 , 19 , 20 , 21, or 22, we obtain that $U^3 \subset \langle z_2u_4, z_2u_5, z_3u_1, z_3u_5, z_4u_1\rangle$ and so forth. □

Type 24: [11, 13, 24, 25]

We put $z_1=u_1^2$, $z_2=u_1u_3$, $z_3=u_2u_4$, and $z_4=u_2u_5$. Omitting the case that is reduced to Type 9 , 10 , 11 , 12 , 13 , 14 , 15 , 19 , 20 , 21, or 22, we obtain that $U^3 \subset \langle z_1u_3, z_2u_4, z_2u_5, z_3u_1, z_4u_1, z_4u_4\rangle$ and that z_1u_3, z_2u_4, z_2u_5, z_3u_1, z_4u_1, z_4u_4 are linearly dependent, so $\dim U^3 \leqq 5$, or else $(U^2)^2=\langle 0\rangle$ and that $\dim U^3 \leqq 6$. □

Type 25: [13, 14, 15, 22]

We put $z_1=u_1u_3$, $z_2=u_1u_4$, $z_3=u_1u_5$, and $z_4=u_2^2$. Omitting the case that is reduced to Type 6 , 7 , 12 , 13 , 14 , 15 , 16 , 19 , 20 , 23, or 24, we obtain that $U^3 \subset \langle z_1u_4, z_1u_5, z_2u_5\rangle$ and so forth. □

Type 26: [12, 13, 14, 15]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, $z_3=u_1u_4$, and $z_4=u_1u_5$. Omitting the case that is reduced to Type 7 , 14 , 16, or 25, we obtain that $U^3 \subset \langle z_1u_3, z_1u_4, z_1u_5, z_2u_4, z_2u_5, z_3u_5\rangle$ and that z_1u_3, z_1u_4, z_1u_5, z_2u_4, z_2u_5 z_3u_5 are linearly dependent, so $\dim U^3 \leqq 5$, or else $(U^2)^2=\langle 0\rangle$ and that $\dim U^3 \leqq 6$. □

Type 27: [12, 13, 14, 25]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, $z_3=u_1u_4$, and $z_4=u_2u_5$. Omitting the case that is reduced to Type 7 , 9 , 13 , 14 , 16 , 17 , 21 , 22 , 23 , 24 , 25, or 26, we obtain that $U^3 \subset \langle z_1u_3, z_1u_4, z_1u_5, z_2u_4\rangle$ and so forth. □

Type 28: [12, 13, 24, 35]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, $z_3=u_2u_4$, and $z_4=u_3u_5$. Omitting the case that is reduced to Type 9 , 13 , 16 , 17 , 21 , 22 , 23 , 24 , or 27, we obtain that $U^3 \subset \langle z_1u_3, z_1u_4, z_2u_5\rangle$ and so forth. □

Type 29: [12, 14, 23, 35]

We put $z_1=u_1u_2$, $z_2=u_1u_4$, $z_3=u_2u_3$, and $z_4=u_3u_5$. Omitting the case that is reduced to Type 9 , 13 , 16 , 17 , 21 , 22 , 23 , 24 or 27, we get that $U^3 \subset \langle z_1u_3, z_1u_4, z_3u_5\rangle$ and so on. □

Type 30: [12, 13, 23, 45]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, $z_3=u_2u_3$, and $z_4=u_4u_5$. Omitting the case that is reduced to Type 2 , 15 , 16 , 21 , 22 , 24 , 27, or 28, we obtain that $U^3 \subset \langle z_1u_3, z_3u_1\rangle$ and so forth. □

Type 31: [11, 22, 34, 56]

We put $z_1=u_1^2$, $z_2=u_2^2$, $z_3=u_3u_4$, and $z_4=u_5u_6$. Omitting the case that is reduced to Type 8 , 10 , 12 , 18 , 19 , 20 , 21, or 23 , we obtain that $U^3 \subset Kz_1u_2$ and so forth. □

Type 32: [11, 12, 34, 56]

We put $z_1=u_1^2$, $z_2=u_1u_2$, $z_3=u_3u_4$, and $z_4=u_5u_6$. Omitting the case that is reduced to Type 9 , 10 , 11 , 13 , 19 , 21 , 22 , 23 , 24, or 31, we obtain that $U^3 \subset Kz_1u_2$ and so forth. □

Type 33: [11, 23, 24, 56]

We put $z_1=u_1^2$, $z_2=u_2u_3$, $z_3=u_2u_4$, and $z_4=u_5u_6$. Omitting the case that is reduced to Type 6 , 12 , 13 , 14 , 15 , 19 , 20 , 21 , 22 , 23 , 24 , 25 , 27 , 29 , 30 , 31, or 32, we obtain that $U^3 \subset Kz_2u_4$ and so forth. □

Type 34: [12, 13, 14, 56]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, $z_3=u_1u_4$, and $z_4=u_5u_6$. Omitting the case that is reduced to Type 7 , 14 , 16 , 21 , 22 , 24 , 25 , 26 , 27 , 29 , 30, or 33, we obtain that $U^3 \subset \langle z_1u_3, z_1u_4, z_2u_4\rangle$ and so forth. □

Type 35: [12, 13, 24, 56]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, $z_3=u_2u_4$, and $z_4=u_5u_6$. Omitting the case that is reduced to Type 9 , 13 , 16 , 17 , 21 , 22 , 23 , 24 , 27 , 28 , 29 , 30 , 32 , 33, or 34, we obtain that $U^3 \subset \langle z_1u_3, z_1u_4\rangle$ and so forth. □

Type 36: [13, 14, 25, 26]

We put $z_1=u_1u_3$, $z_2=u_1u_4$, $z_3=u_2u_5$, and $z_4=u_2u_6$. Omitting the case that is reduced to Type 21 , 22 , 24 , 27 , 29 , 30 , 33 , 34, or 35, we obtain that $U^3 \subset \langle z_1u_4, z_3u_6\rangle$ and so on. □

Type 37: [11, 23, 45, 67]

We put $z_1=u_1^2$, $z_2=u_2u_3$, $z_3=u_4u_5$, and $z_4=u_6u_7$. Omitting the case that is reduced to Type 19, 21, 23, 31, 32, 33, or 35, we obtain that $U^3=\langle 0\rangle$ and so forth. □

Type 38: [12, 13, 45, 67]

We put $z_1=u_1u_2$, $z_2=u_1u_3$, $z_3=u_4u_5$, and $z_4=u_6u_7$. Omitting the case that is reduced to Type 21, 22, 24, 27, 29, 30, 32, 33, 34, 35, 36, or 37, we obtain that $U^3\subset Kz_1u_3$ and so on. □

Type 39: [12, 34, 56, 78]

We put $z_1=u_1u_2$, $z_2=u_3u_4$, $z_3=u_5u_6$, and $z_4=u_7u_8$. Omitting the case that is reduced to Type 32, 35, 37, or 38, we obtain that $U^3=\langle 0\rangle$ and so forth. □

Corollary 4. If $\dim U^2=4$ and $U^2=Z$, then $\dim UZ\leqq 5$ or else $Z^2=\langle 0\rangle$ and, in either case, $\dim UZ\leqq 6$.

Remark on Theorem 5.

One cannot replace the number 6 in the theorem with any other less values. In order to show this, we shall construct the example of Bernstein algebra in which $\dim U^2=4$ and $\dim U^3=6$ (and $(U^2)^2=\langle 0\rangle$).

Example.

Let $A=\langle e, u_1, \cdots, u_{10}, z_1, \cdots, z_4\rangle$ be a commutative 15-dimensional algebra having the following multiplication table:

$$e^2=e \quad eu_i=\frac{1}{2}u_i \quad ez_j=0 \quad u_j^2=z_j \quad (i=1,\cdots,10; j=1,\cdots,4)$$

$$u_1u_2=\alpha z_1+\frac{1}{4}\alpha^{-1}z_2$$

$$u_1u_3=2\alpha\beta z_1+\frac{1}{8}(\alpha\beta)^{-1}\gamma z_3$$

$$u_1u_4=4\alpha\beta\gamma z_1+\frac{1}{16}(\alpha\beta\gamma)^{-1}z_4$$

$$u_2u_3=\beta z_2+\frac{1}{4}\beta^{-1}z_3$$

$$u_2u_4=2\beta\gamma z_2+\frac{1}{8}(\beta\gamma)^{-1}z_4$$

$$u_3u_4=\gamma z_3+\frac{1}{4}\gamma^{-1}z_4$$

$$z_1u_2=u_5 \quad z_1u_3=u_6 \quad z_1u_4=u_7$$

$$z_2u_3=u_8 \quad z_2u_4=u_9 \quad z_3u_4=u_{10}$$

$$z_2u_1=-2\alpha u_5 \quad z_3u_1=-4\alpha\beta u_6$$

$$z_4u_1=-8\alpha\beta\gamma u_7 \quad z_3u_2=-2\beta u_8$$

$$z_4u_2=-4\beta\gamma u_9 \quad z_4u_3=-2\gamma u_{10},$$

where α, β, γ are arbitrary nonzero elements in K, and other products are zero. Then one can see that A is a Bernstein algebra having the decomposition $A=Ke\dot{+}U\dot{+}Z$ with respect to the idempotent e with $U=\langle u_1, \cdots, u_{10}\rangle$, $Z=\langle z_1, \cdots, z_4\rangle$ and, moreover, that it satisfies $U^2=Z$, $U^3=\langle u_5, \cdots, u_{10}\rangle$ and $(U^2)^2=\langle 0\rangle$.

We hope to generalize the relation between $\dim U^2$ and $\dim U^3$ to the case of $\dim U^2>4$. For that purpose it may be more desirable to prove Theorem 3, Theorem 4, and Theorem 5 in rather conceptual method than such computational one as given here.

References

1) A. Wörz-Busekros, *Algebras in Genetics* Springer-Verlag. Berlin-Heidelberg. pp.203–223 (1980)