A note on Bernstein–Jordan Algebras (2)

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Abstract

As a contribution of the problem of classifying all Bernstein algebras, we attempt to describe the possibility of $u$-part decomposition of Bernstein–Jordan algebras in terms of the direct product of Bernstein algebras and study some conditions for a Bernstein–Jordan algebra to be $u$-part decomposable.

§ 1. Preliminaries

Throughout this paper we shall consider commutative algebras of finite dimension over an infinite field $K$ of characteristic not two.

A commutative algebra $A$ over $K$ is called a Bernstein algebra if there exists a nonzero algebra homomorphism $\omega : A \rightarrow K$ ($\omega$ is called weight homomorphism) and every element $x$ in $A$ satisfies the equation

$$(x^2)^2 = \omega(x)x^2.$$ 

The following is a brief description of some known results on Bernstein algebras, which will be used in this study.

(i) For any Bernstein algebra the weight homomorphism is uniquely determined.

(ii) Any Bernstein algebra $A$ has at least one idempotent. For each idempotent $e \in A$, one has $A = Ke \oplus U_e \oplus Z_e$ (direct sum of vector spaces) called the Peirce decomposition of $A$ associated to $e$, where

$$U_e = \{x \in A \mid ex = 1/2x\}, \quad Z_e = \{x \in A \mid ex = 0\}$$

and $\ker \omega = U_e + Z_e$. Then the relations

$$U_eZ_e \subseteq U_e, \quad Z_e^2 \subseteq U_e, \quad U_e^2 \subseteq Z_e$$

(1)

and the identities

$$u^3 = u(uz) = u^2z = (uz)^2 = (u^2)^2 = 0$$

(2)

for every $u \in U_e$ and $z \in Z_e$ hold.

For any given idempotent $e$ of $A$, the set of idempotent elements of $A$ is given by $E(A) = \{e + t + f \mid t \in U_e\}$, and for two idempotents $e$ and $f = e + t + f$, we have the relations

$$U_f = \{u + 2tu \mid u \in U_e\}$$

(3)

$$Z_f = \{z - 2(t + f)z \mid z \in Z_e\}$$

(4)

between the corresponding Peirce spaces, and it is also known that $\dim U_e = \dim U_f$ and $\dim Z_e = \dim Z_f$. A Bernstein algebra $A$ is said to be trivial if $U_e = Z_e = \{0\}$, that is, $A \cong K$. The Peirce spaces $U_e$ and $Z_e$ associated to an idempotent $e$ may be written simply $U$ and $Z$, respectively, when $e$ is fixed.

By definition a commutative algebra $A$ is called a Jordan algebra if the identity $x^2(xy) = (x^2y)x$ holds for every $x, y$ in $A$. We will call a
commutative algebra $A$ a Bernstein–Jordan algebra if $A$ is both Bernstein algebra and Jordan algebra.

(iii) A Bernstein algebra $A = Ke \oplus U_e \oplus Z_e$ with an idempotent $e$ is a Jordan algebra if and only if $z = (uz)z = 0$ for every element $u$ in $U_e$ and $z$ in $Z_e$. Furthermore, a Bernstein algebra $A$ is Jordan if and only if $Z_j^2 = \{0\}$ for every idempotent $f$ of $A$ (cf. [2]).

(iv) If $A$ is a Bernstein–Jordan algebra (of finite dimension), the ideal $\text{Ker} \omega$ is nilpotent, that is, there exists a positive integer $m$ such that $\langle \text{Ker} \omega \rangle^{m+1} = \{0\}$.

(v) If $A = Ke \oplus U_e \oplus Z_e$ is a Bernstein–Jordan algebra, then the subspace $I_e = U_e Z_e \oplus Z_e$ of $A$ is a baric ideal, that is, ideal included in $\text{Ker} \omega$. Moreover, $I_e = I_f$ always holds for any two idempotents $e$ and $f$ of $A$ (cf. [3]). We denote this ideal by $I$. The factor space $A/I$ is a Bernstein–Jordan algebra and the weight homomorphism $\omega$ of $A/I$ is defined by $\overline{\omega}(\phi(x)) = \omega(x)$ for any element $x$ in $A$, where $\phi$ is the canonical homomorphism of $A$ onto $A/I$.

§ 2. Decomposability of Bernstein–Jordan algebras

As shown by T. Cortés and F. Montaner (cf. [1]), the direct product of a family of Bernstein algebras $(A_i, \omega_i)_{1 \leq i \leq n}$ is a Bernstein algebra $X_{1 \leq i \leq n} A_i$, which is defined as a set by

$$X_{1 \leq i \leq n} A_i := \{ (x_i) \in \prod_{1 \leq i \leq n} A_i | \omega_i(x_i) = \omega_j(x_j) \text{ for all } i, j \}$$

and, as an algebra, is a subalgebra of $\Pi_{1 \leq i \leq n} A_i$ equipped with the weight homomorphism $\omega$ defined by $\omega((x_i)) = \omega_j(x_j)$ for any $j (1 \leq j \leq n)$. This implies that for any given Bernstein algebra $C$ with the weight homomorphism $\tau$ and homomorphisms $\varphi_i : C \rightarrow A_i$ such that $\tau = \omega_i \varphi_i$, for all $i (1 \leq i \leq n)$, there always exists the unique homomorphism $\varphi : C \rightarrow X A_i$ such that $\omega \varphi = \tau$.

Then, by definition, a Bernstein algebra $A$ is decomposable if there exist nontrivial Bernstein algebras $A_1$ and $A_2$ such that $A \cong A_1 \times A_2$ and indecomposable if it is not decomposable.

It is known that the decomposability of Bernstein algebras is characterized as follows (cf. [1]).

Proposition A. A Bernstein algebra $A$ is decomposable if and only if there exist nonzero baric ideals $I_1$, $I_2$ of $A$ such that $\text{Ker} \omega = I_1 \oplus I_2$ as algebras. In this case $A \cong A/I_1 \times A/I_2$.

Proposition B. A Bernstein algebra $A$ is decomposable if and only if there exist nontrivial subalgebras $A_1$, $A_2$ of $A$ such that $(A_1 \cap \text{Ker} \omega) \langle A_2 \cap \text{Ker} \omega \rangle = \{0\}$, $A_1 + A_2 = A$, and $A_1 \cap A_2$ is a trivial algebra. In this case $A \cong A_1 \times A_2$.

Remark 1. The ideals $I_i$ and subalgebras $A_i$ in propositions stated above are related by

$$A/I_i \cong A_2, \ A/I_i \cong A_1, \ A_1 = Ke + I_i.$$ 

T. Cortés and F. Montaner reduced the problem of classifying all Bernstein algebras to the case of indecomposable Bernstein–Jordan algebras, while they found that there exist infinite families of indecomposable Bernstein–Jordan algebras (cf. [1]). Therefore it is expected that the decomposition of indecomposable Bernstein–Jordan algebras into Bernstein–Jordan algebras of more simple type.

Now we shall define $u$-part decomposability for Bernstein–Jordan algebras.

Definition 2. Let $A$ be a Bernstein–Jordan algebra $A$ such that $U_e = \langle 0 \rangle$ for any idempotent $e$ of $A$. If there exist nonzero subspaces $U_1$ and $U_2$ of $U_e$ such that

$$U_e = U_1 \oplus U_2,$$

for $i = 1, 2$, then $A$ is said $u$-part decomposable.

We can prove that $u$-part decomposability is not dependent on the choice of idempotent $e$ (cf.
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Let $A = Ke \oplus U \oplus Z$ be the Peirce decomposition of a Bernstein algebra $A$ associated to some idempotent $e$ and $\phi : A \to A/I$ be the canonical homomorphism where the ideal $I = UZ \oplus Z$ (see §1(v)). Then we can put $\phi (x) \equiv ae + u (\text{mod } I)$ for all $x = ae + u + z \in A$.

Lemma 3.
(a) For the factor algebra $A/I$, there exists an isomorphism $A/I \cong Ke \oplus U/UE$ as spaces.
(b) If $A$ is $u$-part decomposable and $U_1$, $U_2$ are both nonzero subspaces of $U$ satisfying the condition (5), then $U/UE \cong U_1/U_1Z \oplus U_2/U_2Z$ as spaces.

Proof. (a) Since $I = UZ \oplus Z$ is a baric ideal of $A$ and $UZ$ is a subspace of $U$, $A/I \cong Ke \oplus U/UE \oplus Z/Z \cong Ke \oplus U/UE$.

(b) Since $UZ \subset U_1$ and $U_1 \cap U_2 = \langle 0 \rangle$, we have $UZ \cap U_1Z = \langle 0 \rangle$, so $UZ = U_1Z \oplus U_2Z$. Therefore $U/UE \cong U_1/U_1Z \oplus U_2/U_2Z$. □

Remark 4. If we define spaces $A_i = Ke \oplus U_i \oplus Z$ and $\tilde{A}_i = A_i/(A_i \cap I)$ $(i = 1, 2)$, then, as easily seen, they are Bernstein algebras with weight homomorphisms $w_i = w \mid A_i$ and $\tilde{w}_i = w \mid \tilde{A}_i$, respectively, and $\tilde{A}_i \cong Ke \oplus U_i/(U_iZ)$ as spaces.

In the following discussion we assume that algebra $A$ is a Bernstein–Jordan algebra with the weight homomorphism $\omega$ such that $\dim U > 0$. Let $A = Ke \oplus U \oplus Z$ be the Peirce decomposition of $A$ associated to some idempotent $e$. We recall that $I$ is the (baric) ideal defined by $I = UZ \oplus Z$.

We assert that, if $A$ is $u$-part decomposable, then the factor algebra $A/I$ is decomposable.

Proposition 5. Let $A$ be $u$-part decomposable and $U_i (i = 1, 2)$ nonzero subspaces such that $U = U_1 \oplus U_2$ and $U_i \subset U_i$, and $A_i$ Bernstein–Jordan algebras defined by $A_i = Ke \oplus U_i \oplus Z$ $(i = 1, 2)$. Then $\tilde{A} = A_1 \times A_2$, where $\tilde{A} = A/I$ and $\tilde{A}_i = A_i/(A_i \cap I)$ $(i = 1, 2)$.

Proof. For $i = 1, 2$ let $\pi_i$ be the canonical projection of $U$ onto $U_i$ and define the mapping $\sigma_i$ of $\tilde{A}$ to $\tilde{A}_i$ by $\sigma_i(\phi(x)) = ae + \phi(\pi_i(u))$ for any $x = ae + u + z \in A$, where $\phi$ is the canonical homomorphism $A \to \tilde{A}$. Then we can see easily that $\sigma_i(i = 1, 2)$ are well-defined homomorphisms. We show that $\sigma_i$ satisfies $\tilde{w}_i = \tilde{w}_i \sigma_i$ for $i = 1, 2$. For any element $x = ae + u + z \in A$ we have $	ilde{w}(\phi(x)) = \omega(x) = \tilde{w}_i(\phi(x)) = \tilde{w}_i((ae + \phi(\pi_i(u)))) = \pi_i((ae + \phi(\pi_i(u)))) = \pi_i(ae + \phi(\pi_i(u))) = \pi_i(ae + \phi(\pi_i(u))) = a$. Hence $\tilde{w} = \tilde{w}_i \sigma_i$.

We define the mapping $\sigma$ of $\tilde{A}$ to $\tilde{A}_1 \times \tilde{A}_2$ by $\sigma(\phi(x)) = (\sigma_1(\phi(x)), \sigma_2(\phi(x)))$ for each $x \in A$. Then we can prove that $\sigma$ gives an isomorphism of $\tilde{A}$ onto $\tilde{A}_1 \times \tilde{A}_2$ as follows.

First we note that $\sigma$ is well-defined since $\tilde{w}_i \sigma_i(\phi(x)) = \tilde{w}_i \sigma_i(\phi(x)) = \omega(x)$ for any $x \in A$. By easy calculation we see that $\sigma$ is a homomorphism of algebras. Let $\sigma(\phi(x)) = 0$ for $x = ae + u + z \in A$. Then $\sigma(\phi(x)) = \sigma(\phi(x)) = 0 \equiv 0(\text{mod } I)$, so $\sigma(\phi(x)) = \sigma(\phi(x)) = 0 \equiv 0(\text{mod } I)$ and $\sigma(\phi(x)) = \sigma(\phi(x)) = 0 \equiv 0(\text{mod } I)$, which means that $\sigma$ is a monomorphism. It is trivial that $\sigma$ is a homomorphism.

We concern on the inverse problem of Proposition 5.

Assume that $\tilde{A} = A/I$ with $I = UZ \oplus Z$ and $\tilde{A}$ is decomposable. Then, by Proposition A, there exist nonzero baric ideals $I_i (i = 1, 2)$ of $\tilde{A}$ such that $\ker \omega = I_1 \oplus I_2$ and in this case $\tilde{A} \cong \tilde{A}_1 \times \tilde{A}_2$. Let $\phi_i : A \to \tilde{A}_i (i = 1, 2)$ be the canonical homomorphisms, $\tau_i$ the weight homomorphism of $\tilde{A}/I_i$ defined by $\tau_i(\phi_i(\phi(x))) = \tilde{w}_i(\phi(x))$ and $\tau$ the weight homomorphism of $\tilde{A}/I_1 \times \tilde{A}/I_2$ defined by $\tau(\phi_1(\phi(x)), \phi_2(\phi(x))) = \tau(\phi_1(\phi(x)))$.

Then the isomorphism $\phi : \tilde{A} \cong A/I_1 \times A/I_2$ is defined by

$\phi(\phi(x)) = (\tau(\phi_1(\phi(x)), \phi_2(\phi(x))) \quad \text{for } \phi(x) \in A$.

Moreover define the epimorphisms $\phi_i : A \to \tilde{A}_i$.
Lemma 6. The space $I_i$ satisfies $I_i = \ker \phi_i$ for $i = 1, 2$.

Proof. Let $x$ be any element of $A$. Since $\phi_i(x) = \phi(\phi_i(x) \equiv \phi(x) \mod I_i)$, the condition $x \in \ker \phi_i$ is equivalent to the condition $\phi_i(x) \equiv 0 \mod I_i$, i.e. $x \in I_i$. \qed

Lemma 7. The space $I_i$ is a baric ideal of $A$ satisfying $I_i Z \subseteq I_i$ for $i = 1, 2$.

Proof. For any $x \in I_i$ and $z \in Z$ we obtain that $\phi(xz) \equiv \phi(x) \phi(z) \mod I_i$, which implies $I_i Z \subseteq I_i$. We note that $\phi_i(I_i) = \overline{I_i}$, so $\phi_i(I_i) \subseteq I_i$, which means that $I_i \subseteq J_i$. Similarly $I_2 \subseteq J_2$ is obtained. \qed

Now define the subspaces $V_i (i = 1, 2)$ of $U$ by

$$V_i = I_i \cap U.$$

Lemma 8. We have the following relations.
(a) $I_i = V_i \oplus Z (i = 1, 2)$
(b) $V_i Z \subseteq V_i (i = 1, 2)$
(c) $V_i \cap V_2 = U Z$
(d) $U = V_i + V_2$

Proof. (a) For any element $z$ of $Z$ and $i = 1, 2$,

$$\phi_i(z) = \phi(\phi(z) \equiv \phi_i(0) \equiv 0 \mod I_i) \equiv 0 \mod I_i$$

so $z \in \ker \phi_i$. This implies $Z \subseteq I_i$ by Lemma 6. Hence $I_i = I_i \cap (U \oplus Z) = I_i \cap U \oplus I_i \cap Z$ and $I_i \cap Z = Z$. By definition of $V_i$, we obtain $I_i = V_i \oplus Z$ as spaces. (b) $V_i Z = (I_i \cap U) Z \subseteq I_i Z \subseteq U Z \subseteq I_i \cap U = V_i$ by Lemma 7. (c) Let $x$ be any element of $V_i \cap V_2$. From the definition of $V_i$, $x \in U \cap V_i \cap V_2$. Then we have $\phi(x) \equiv \overline{I_i} \cap \overline{I_2}$ since $\phi_i(x) \equiv \phi(x) \equiv 0 \mod I_i$ for $i = 1, 2$ by Lemma 6. Therefore $\phi(x) \equiv 0 \mod I_i$, so $x \in I_i$. On the otherhand, since $x \in U$ and $u$-part of $I$ is $U Z$, we have $x \in U Z$. As a result we obtain $V_i \cap V_2 \subseteq U Z$. The inverse inclusion is easy. (d) For each $u \in U$ there exist uniquely elements $u_1, u_2$ of $I_i (i = 1, 2)$ such that $\phi(u) = u_1 + u_2$, since $\phi_i(\phi_i(u)) = \phi_i(u) \equiv 0$. Then we can take elements $u_i \in I_i (i = 1, 2)$ such that $\phi_i(u_i) = u_i$. Since for all elements $z$ of $Z$ $\phi_i(u_i + z) = \phi_i(u_i)$, we can suppose $u_i \in U (i = 1, 2)$. Hence $u_i \in V_i$. Then by the way of choice of $u_i$, we have $\phi_i(u - u_1 - u_2) = \phi_i(u) - \phi_i(u_1) - \phi_i(u_2) \equiv 0 \mod I_i$, so $u - u_1 - u_2 \in I_i$. Hence $u - u_1 - u_2 \in I \cap U = U Z$, so $u \in U Z + V_i + V_2$. This means that $U \subseteq U Z + V_i + V_2$. On the otherhand $U Z \subseteq V_i + V_2$ from the result of (c) above. Consequently we obtain $U \subseteq V_i + V_2$. The inverse inclusion is obvious. \qed

For each $i = 1, 2$ there exists a subspace $W_i$ of $V_i$ such that

$$V_i = U Z \oplus W_i.$$

For such $W_i$ we define the subspace $U_i$ of $U$ by

$$U_i = W_i + \sum_{k \geq i} W_i Z^{(k)}$$

where $W_i Z^{(k)}$ is defined by $W_i Z^{(k)} = W_i Z$.

Lemma 9. We have the following relations.
(a) $U_i \subseteq V_i$ and $U_i Z \subseteq U_i (i = 1, 2)$
(b) $U_i = W_i \oplus \sum_{k \geq i} W_i Z^{(k)}$
(c) $U = U_1 + U_2$
(d) $U Z = \sum_{k \geq 1} W_i Z^{(k)} + \sum_{k \geq 1} W_i Z^{(k)}$

Proof. (a) Since $W_i \subseteq V_i$, $U_i \subseteq V_i$ and $U_i Z \subseteq U_i$. (b) From $W_i Z^{(k)} \subseteq U Z$, it follows that $\sum W_i Z^{(k)} \subseteq U Z$. Since $U Z \cap W_i = \{0\}$, it follows that $\sum W_i Z^{(k)} \cap W_i = \{0\}$. (c) From the equation $U = V_1 + V_2 = U Z + W_1 + W_2$ we obtain by induction $U \subseteq U Z^{(m)} + U_1 + U_2$ for all integer $m \geq 1$. When $m$ is large enough, $U Z^{(m)} = \{0\}$ (cf. §1(iv)). Therefore...
fore \( U \subseteq U_1 + U_2, \) so \( U = U_1 + U_2. \) (d) By the similar fashion as the proof of (c) above we have \( UZ \subseteq UZ^{(m)} = \Sigma W_i Z^{(k)} \) for every integer \( m \geq 1. \) Thus we obtain \( UZ \subseteq \Sigma W_i Z^{(k)} \) for \( \Sigma W_i Z^{(k)} \). The inverse inclusion is obvious. \( \square \)

**Proposition 10.** If \( \Sigma_{k=1}^m W_i Z^{(k)} \cap \Sigma_{k=1}^m W_i Z^{(k)} = \langle 0 \rangle, \) then the spaces \( U_i = W_i + \Sigma_{k=1}^m W_i Z^{(k)} \) for \( i = 1, 2 \) satisfies that \( U = U_1 \oplus U_2 \) and \( UZ \subseteq U_i \), that is, a Bernstein-Jordan algebra \( A \) is \( u \)-part decomposable.

**proof.** From the results of Lemma 8 (d) and Lemma 9 (d) we see that \( U = V_1 + V_2 = UZ + W_1 + W_2 = \Sigma W_i Z^{(k)} + \Sigma W_i Z^{(k)} + W_1 + W_2. \) We can show directly that \( \Sigma W_i Z^{(k)} + \Sigma W_i Z^{(k)} + W_1 + W_2 \) is direct sum of spaces as follows. By hypnosis of the proposition, \( \Sigma W_i Z^{(k)} \cap \Sigma W_2 Z^{(k)} = \langle 0 \rangle. \) Since \( UZ \cap W_i = \langle 0 \rangle \), \( \Sigma W_i Z^{(k)} \cap \Sigma W_2 Z^{(k)} \cap W_i = \langle 0 \rangle. \) Then, since \( V_1 \cap W_i \subseteq UZ \) by Lemma 8 (c) and \( UZ \cap W_2 = \langle 0 \rangle, \) it follows that \( \Sigma W_i Z^{(k)} \cap \Sigma W_2 Z^{(k)} \cap W_i = \langle 0 \rangle \). These imply that \( \Sigma W_i Z^{(k)} + \Sigma W_i Z^{(k)} + W_1 + W_2 = \Sigma W_i Z^{(k)} \cap \Sigma W_2 Z^{(k)} \cap W_i \cap W_2 = \langle 0 \rangle. \) Hence \( U = (W_1 \oplus W_2) \oplus (W_1 \oplus W_2) \).

**Corollary 11.** If \( V_1 Z \cap V_2 Z = \langle 0 \rangle, \) then \( A \) is \( u \)-part decomposable.

**proof.** Take a subspace \( W_i \) such that \( V_i = UZ \oplus W_i \) for \( i = 1, 2. \) By Lemma 8 (b), \( W_i Z \subseteq V_i Z \subseteq V_i. \) Therefore, by induction, \( W_i Z^{(k)} \subseteq V_i Z \) for each integer \( k \geq 1, \) so \( \Sigma W_i Z^{(k)} \subseteq V_i Z. \) Hence \( \Sigma W_i Z^{(k)} \cap \Sigma W_2 Z^{(k)} \subseteq V_i Z \cap V_2 Z = \langle 0 \rangle. \) The rest of proof is obvious from Proposition 10. \( \square \)

**Theorem 12.** Assume that \( A = Ke + U + Z \) is a nontrivial Bernstein-Jordan algebra and \( I \) is the ideal of \( A \) defined by \( I = UZ \oplus Z \) such that \( \overline{A} = A/I \) is decomposable with \( F = \langle 0 \rangle. \) Then \( A \) is \( u \)-part decomposable, that is, there exist nonzero subspaces \( U_i (i = 1, 2) \) such that \( U = U_1 \oplus U_2 \) and \( UZ \subseteq U_i. \)

**proof.** First we observe that the condition \( F = \langle 0 \rangle \) is equivalent to that \( (UZ)Z = \langle 0 \rangle \) since \( Z = \langle 0 \rangle \) and \( (UZ)^2 = \langle 0 \rangle \) by the assumption and the properties of Bernstein-Jordan algebras. Now we take the bases \( \{x_i, y_i\}_{1 \leq i \leq s}, \{x_i, w_k\}_{1 \leq i \leq s, 1 \leq k \leq t} \) of \( V_1 \) and \( V_2 \) respectively such that \( \{x_i\}_{1 \leq i \leq s} \) is the basis of \( UZ \), and \( \{y_i\}_{1 \leq i \leq s} \) and \( \{w_k\}_{1 \leq k \leq t} \) are the bases of \( W_1 \) and \( W_2 \) respectively, where \( r = \dim UZ \), \( s = \dim W_1 \), and \( t = \dim W_2. \) Then we can assume that there exists at least one base element \( v \) in the set \( B = \{v_i, w_k\}_{1 \leq i \leq s; 1 \leq k \leq t} \) such that \( vZ \neq \langle 0 \rangle. \) Because, if it is not so, \( V_1 Z = V_2 Z = \langle 0 \rangle \) from \( (UZ)Z = \langle 0 \rangle \), so the proof is reduced to Corollary 11. We separate the set \( B \) into two nonempty sets \( B_1 = \{y \in B | \ yZ \neq \langle 0 \rangle\} \), \( B_2 = \{y \in B | \ yZ = \langle 0 \rangle\} \) and construct subspaces \( V_i, (i = 1, 2) \) of \( U \), where \( V_i (i = 1, 2) \) are respectively defined as the spaces spanned by the sets \( \{x_i\}_{1 \leq i \leq s} \cup B_i. \) Then, as easily seen, \( U = V_1 \oplus V_2, V_1 Z \subseteq V_1 \cap V_2 = UZ \) and \( V_2 Z \subseteq V_2 \cap V_1 = UZ \). Hence the proof is reduced to Corollary 11 again. \( \square \)

**Example 13.** As an example of indecomposable but \( u \)-part decomposable algebra, we take up the Bernstein-Jordan algebra \( A = A(\alpha, \beta) \) for \( \alpha, \beta \in K \), with basis \( e, u_1, u_2, u_3, z_1, z_2 \) and multiplication table \( e^2 = e, \ e u_i = 1/2 u_i, \ e z_1 = 0, \ u_i^2 = z_1 + z_2, \ u_i^2 = z_1 + z_2, \ u_i z_2 = z_1 + \beta z_2, \ u_i u_i = 0 \) for \( i \neq j \) and \( z_2 z_2 = u_j z_2 \) for some \( j \). It is known that this algebra \( A \) is indecomposable and that there exist infinitely many nonisomorphic algebras of the form \( (\alpha, \beta) \) (cf. [1]). In this case the ideal \( I = UZ + Z \) is equal to \( Z = Kz_1 + Kz_2 \) since \( UZ = \langle 0 \rangle \) by multiplication table. Moreover \( \overline{A} = A/I \cong Ke \oplus I_1 \oplus I_2 \oplus I_3 \oplus I_4 \), where \( I_1 \cong K u \) are nonzero baric ideals. Hence \( A \) is decomposable by Proposition A. Therefore \( A \) is \( u \)-part decomposable by Theorem 11 since \( F = \langle 0 \rangle \).

**References**


