On Bernstein Algebras, II

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Introduction

In this paper, we are concerned with the characterization of Bernstein algebras from the viewpoint of the existence of certain idempotents and with the structure of a Bernstein algebra based on the information upon the idempotents in it.

A commutative algebra $A$ over a field $K$ is called a Bernstein algebra if there exists a non-trivial algebra homomorphism $\omega$ of $A$ into $K$ and the identity $(x^2)^2 = \omega(x)^2x^2$ holds in $A$. It is known that, in a Bernstein algebra, the homomorphism $\omega$ is uniquely determined and there exists at least one idempotent $e (0 \neq e = e^2)$.

In general an algebra equipped with a non-trivial algebra homomorphism onto its base field (called weight homomorphism) is said to be baric. In view of that, it is natural to try to describe how Bernstein algebras should be characterized as the commutative baric algebras which have the idempotents imposed a certain condition on.

In that respect, we shall see that a commutative baric algebra $A$ with weight homomorphism $\omega$ is a Bernstein algebra if and only if, in $A$, there exists at least one idempotent $e$ such that $x = e$ satisfies the equation $(xa)(bc) + (xb)(ac) + (xc)(ab) = (xa)^\circ(bc) + (xb)^\circ(ac) + (xc)^\circ(ab)$ for all $a, b, c \in A$, where the operator $^\circ$ is defined by $x^\circ y = \frac{1}{4}(\omega(x)y + \omega(y)x)$. Notice that, in case of char $K \neq 2, 3$, this equation is equivalent to $(x^2)^2 = \omega(x)^2x^2$.

As well known, a Bernstein algebra $A$ splits into the direct sum $A = Ke \oplus U \oplus Z$ (the Peirce decomposition with respect to an idempotent $e$), where $U : = \{x \in A \mid ex = 1_4 x\}, Z : = \{x \in A \mid ex = 0\}$. Denote by $V$ the subspace $U^\circ$.

Holgate, P. has noticed in his treatise [2] on Bernstein algebras that a large number of cases in those particular Bernstein algebras examined by him are the case of $VZ = 0$ as established by himself and left it as an open problem to establish the condition under which $VZ = 0$. To this problem we shall apply the approach devised by S. González and C. Martínez[1] to establish one necessary and sufficient condition to be $VZ = 0$. More precisely speaking, we shall show that $VZ = 0$ for each idempotent $e$ if and only if, for each pair of idempotents, $e, f$, the assertion $e\rho f$ implies $e = f$, where the symbol $\rho$ denotes the equivalence relation on all the idempotents of $A$ defined by relating $e$ and $f$, $e\rho f$ if $V = V_f$ (the $V$-part of the Peirce decomposition with respect to $f$) and there exists $w \in VZ$ such that $f = e + w$.

1. Characterization of Bernstein algebras

Let $A$ be a commutative baric algebra. By definition $A$ is a commutative algebra over a field $K$ such that there exists a non-trivial algebra homomorphism $\omega$ (called weight homomorphism) of $A$ into $K$ (will be assumed to be of characteristic different from 2 through this paper).

**Definition.** A commutative baric algebra $A$ with
weight homomorphism \( \omega \) is called a Bernstein algebra if the identity
\[
(1.1) \quad (x^2)^2 = \omega(x)^2 x^2
\]
holds in \( A \).

Let \( A \) be a Bernstein algebra. Then it is known (cf. [4], Corollary 9.5, Theorem 9.6) that the following statements hold:

There exists at least one idempotent in \( A \) and with respect to each idempotent \( e \) (i.e., \( e^2 = e \neq 0 \)) \( A \) splits into the vector space direct sum (so called the Peirce decomposition of \( A \) with respect to \( e \))
\[
(1.2) \quad A = Ke \oplus U \oplus Z
\]
where
\[
U := \{ x \in A \mid ex = \frac{1}{2} x \}, \quad Z := \{ x \in A \mid ex = 0 \},
\]
and for all \( u \in U \) and \( z \in Z \):
\[
(1.3) \quad \omega(e) = 1, \quad \omega(u) = \omega(z) = 0,
(1.4) \quad u^2 = 0,
(1.5) \quad u(uz) = 0,
(1.6) \quad u^2 z^2 = 0,
(1.7) \quad u^2 (uz) = 0,
(1.8) \quad (uz) z^2 = 0,
(1.9) \quad (uz^2)^2 = 0,
(1.10) \quad u^2 z^2 = 0,
\]
and
\[
(1.11) \quad UZ \subseteq U,
(1.12) \quad U^2 \subseteq Z,
(1.13) \quad Z^2 \subseteq U,
(1.14) \quad UZ^2 = 0.
\]

We shall try to characterize a Bernstein algebra as the commutative baric algebra which has the idempotents imposed a certain condition on. Whenever \( A \) is a commutative baric algebra with weight homomorphism \( \omega \), the following product defines a new multiplication in \( A \):
\[
(1.15) \quad x \ast y = \frac{1}{2} (\omega(x)y + \omega(y)x).
\]

Particularly, if \( A \) is a Bernstein algebra, then the linearization process of the identity (1.1) leads to the identity
\[
(1.16) \quad (xa)(bc) + (xb)(ac) + (xc)(ab)
= (xa) \ast (bc) + (xb) \ast (ac) + (xc) \ast (ab)
\]
for all \( x, a, b, c \in A \), represented with the product \( x \ast y \) (cf. [3]). Reciprocally, it is easily seen that the identity (1.16) implies (1.1) under the assumption \( \text{char} K \neq 3 \).

Again, if \( A \) is a commutative baric algebra and there exists an idempotent \( e \) in \( A \), then for each \( \lambda \in K \) the subspace \( A_e(\lambda) \) of \( A \) can be defined by
\[
(1.17) \quad A_e(\lambda) = \{ x \in A \mid ex = \lambda x \}.
\]

In the case of a Bernstein algebra \( A \) having an idempotent \( e \), \( A_e(1) = 0 \) for every \( \lambda \in K \) except for \( A_e(1) = Ke, A_e(\frac{1}{2}) = U \) and \( A_e(0) = Z \), and the decomposition (1.2) for \( A \) with respect to \( e \) can be written in more general notation;
\[
(1.18) \quad A = A_e(1) \oplus A_e(1/2) \oplus A_e(0).
\]

**Theorem 1.** Let \( A \) be a commutative baric algebra over \( K \) of characteristic \( \neq 2,3 \) with weight homomorphism \( \omega \). Then \( A \) is a Bernstein algebra if and only if there exists an idempotent \( e \) in \( A \) such that \( x = e \) satisfies the equation (1.16) for all \( a, b, c \in A \), i.e., there holds
\[
(1.19) \quad (ea)(bc) + (eb)(ac) + (ec)(ab)
= (ea) \ast (bc) + (eb) \ast (ac) + (ec) \ast (ab)
\]
for all \( a, b, c \in A \).

**Proof.** Suppose that \( A \) is a Bernstein algebra, then the existence of idempotents in \( A \) is guaranteed as stated already and any idempotent \( e \) of \( A \) obviously satisfied the equation (1.19). Reciprocally, suppose that \( A \) has an idempotent \( e \) satisfying the condition (1.19). Then if the following properties (A), (B) and (C) hold, \( A \) is a Bernstein algebra on account of the already known theorem (cf. [4], Theorem 9.7):

(A) \( \omega(e) = 1 \),
(B) With respect to \( e \) the decomposition (1.18) for \( A \) holds, and

(Miyamoto)
Therefore in order to show that $A$ is a Bernstein algebra, it suffices to see that (A), (B) and (C) hold and this will be done in the following lemma.

**Lemma 2.** Let $A$ be a commutative baric algebra with an idempotent $e$ satisfying (1.19). Then the properties (A), (B) and (C) all hold.

**Proof.** (i) By applying $\omega$ to the equation $e^2 = e$ and putting $a = b = c = e$ in (1.19), we obtain $\omega(e)^2 = \omega(e)$ and $3e = 3\omega(e)^2 e$ respectively. From these two equations we easily see that (A) holds.

(ii) In the equation (1.19), put $a = b = e$ and let $c$ be an arbitrary element in $A_{e\ominus}$. Then we obtain $3e^2 (ec) = 3e \omega(ec)$, and so $c = \frac{1}{3} \omega(ec) = \frac{1}{3} c + \frac{1}{3} \omega(c)e$ by virtue of char $K \neq 2$. Thus (B) is fulfilled by $e$ and therefore $c$ $\in$ $Ke$. Consequently we get $A_{e\ominus}(1) = Ke$. Suppose that $\lambda \neq 1$, $\lambda \in K$ and that $x$ is an arbitrary element in the subspace $A_{e\ominus}(1)$ defined by (1.17) for a fixed $\lambda$. Then by applying $\omega$ to the equation $ex = \lambda x$, we obtain $\omega(x) = \lambda \omega(x)$, and hence $\omega(x) = 0$ since $\lambda \neq 1$. Put $a = b = e$ and $c = x$ in (1.19) and calculate using $\omega(x) = 0$. As a result we obtain that $\lambda = 0$ or $\frac{1}{\lambda}$, which implies (B).

(iii) First we remark that $\omega(u) = \omega(z) = 0$ for all $u \in A_{e\ominus}\left(\frac{1}{\lambda}\right) = U$ and $z \in A_{e\ominus}(0) = Z$ since $\omega(u) = \omega(eu) = \frac{1}{\lambda} \omega(u)$ and $\omega(z) = \omega(ez) = \omega(0) = 0$ respectively. In order to prove the identity (1.4), substitute $u \in U$ for all of $a$, $b$ and $c$ in (1.19). Then we obtain the equation (1.4) by using the above remark. The rest identities of (1.4), (1.10) will be obtained individually in the same way as above, which we omit. Thus (C) holds.

In this connection we shall make mention of Jordan-Bernstein algebras. A commutative algebra $A$ is called a Jordan algebra if and only if the so-called Jordan identity $x^2(\langle xy \rangle x) = x(\langle x^2y \rangle x)$ holds in $A$. Under the assumption of char $K \neq 2$ the Jordan identity is equivalent to the following identity:

$$
(1.20) \quad \langle xa \rangle \langle bc \rangle + \langle xb \rangle \langle ac \rangle + \langle xc \rangle \langle ab \rangle = c[b(ax)] + x[b(ac)] + a[b(cx)]
$$

for all $x, a, b, c \in A$.

A Jordan-Bernstein algebra is, by definition, a commutative algebra which is Jordan as well as Bernstein.

**Theorem 3.** Let $A$ be a commutative baric algebra with weight homomorphism $\omega$. Then $A$ is a Jordan-Bernstein algebra if and only if, in $A$ there exists an idempotent $e$ satisfying the following two equations

$$
(1.21) \quad \langle ea \rangle \langle bc \rangle + \langle eb \rangle \langle ac \rangle + \langle ec \rangle \langle ab \rangle = c[b(\langle ea \rangle)] + e[b(\langle ac \rangle)] + a[b(\langle ec \rangle)]
$$

and

$$
(1.22) \quad [b(\langle ea \rangle)] + e[b(\langle ac \rangle)] + a[b(\langle ec \rangle)] = (\langle ea \rangle) \ast (\langle bc \rangle) + \langle eb \rangle \ast (\langle ac \rangle) + (\langle ec \rangle) \ast (\langle ab \rangle)
$$

for all $a, b, c \in A$.

**Proof.** One implication is clear, since in a Jordan-Bernstein algebra each idempotent $e$ satisfies (1.21) and so (1.22) in view of Theorem 1.

Conversely suppose $A$ be a commutative baric algebra, in which there exists an idempotent $e$ satisfying both (1.21) and (1.22). Then the equation (1.19) is fulfilled by $e$ and therefore, by Theorem 1, $A$ is a Bernstein algebra. Consequently, as already stated, with respect to $e$, $A$ splits into the vector space direct sum $A = Ke \oplus U \oplus Z$ with $U: = \{x \in A \mid ex = \frac{1}{\lambda}x\}$ and $Z: = \{x \in A \mid ex = 0\}$. In order to see that $A$ is Jordan, first let us recall that a Bernstein algebra $A = Ke \oplus U \oplus Z$ is a Jordan algebra if and only if $z^2 = 0$ and $\langle uz \rangle z = 0$ for every $u \in U$ and $z$ in $Z$ (cf. [3], Theorem 4; [1], Theorem 1). Next, in (1.21), by putting $a = e$ and $b = c = z$ for an arbitrary $z$ in $Z$ and noticing $ez = 0$, $ez^2 = \frac{1}{\lambda}z^2$ (cf. (1.13)), we obtain $z^2 = 0$. Then in (1.22), by putting $a = u$ and $b = c = z$ for arbitrary $u$ in $U$ and $z$ in $Z$ and noticing $ez = 0$, $eu = \frac{1}{\lambda}u$, etc.,
2. Equivalence relation in the set of idempotents

Let $A$ be a Bernstein algebra of characteristic not 2 with weight homomorphism $\omega$. Denote by $I(A)$ the set of idempotents of $A$.

S. Gonzalez and C. Martinez in [1] studied the set $I(A)$ and tried to transfer the information about $I(A)$ to the whole $A$. They introduced a certain equivalence relation in $I(A)$ and obtained some useful information about $A$ by using it. In this approach it seems interesting for us to find any new equivalence relations in $I(A)$.

In this section we shall consider the Peirce decompositions of $A$ with respect to various idempotents $e$. To distinguish between them we shall adopt the following notations for the components $U$ and $Z$ in the Peirce decomposition corresponding to $e$:

$$U_e := \{x \in A \mid ex = \frac{1}{2}x\}, \quad Z_e := \{x \in A \mid ex = 0\}$$

for each $e \in I(A)$.

Now we summarize some already established observations on idempotents of $A$, which will be used after.

Observations.

(a) If one idempotent $e$ of $A$ is fixed, then $I(A)$ is given by

$$I(A) = \{e + u + u^2 \mid u \in U\}$$

(cf. [4], Lemma 9.8).

(b) If $e_1$ and $e_2$ are two idempotents of $A$, then there are uniquely determined $u_i \in U_i := U_{e_i}$, $i = 1, 2$, such that

$$e_1 = e_2 + u_2 + u_2^2, \quad e_2 = e_1 + u_1 + u_1^2.$$  

Furthermore

$$e_1 e_2 = e_1 + \frac{1}{2} u_1 = e_2 + \frac{1}{2} u_2,$$

and

$$u_1^2 = u_2^2 = -u_1 u_2$$

(cf. [4], Lemma 9.9). Directly from (2.2) and (2.3), we obtain

$$u_1 u_2 = \frac{1}{2} (u_1 + u_2).$$

(c) We shall write $U_{e_0} = \{u \in U_e \mid u U_e = 0\}$ for each idempotent $e \in I(A)$.

Then $U_{e_0} = U_{f_0}$ for every pair of idempotents $e, f \in I(A)$ (cf. [1]). Thus $U_{e_0}$, defined by $U_{e_0} := U_{e_0}$ for some idempotent $e$, is independent of the idempotent $e$.

(d) If $f = e + u_e + u_e^2$ with $u_e \in U_{e_0}$, the relation between $U_{e_0}$ and $U_{f_0}$ and $Z_{e_0}$ and $Z_{f_0}$ for idempotents $e, f \in I(A)$ is given by:

$$U_{f_0} = \{u + 2u_0 u \mid u \in U_{e_0}\},$$

$$Z_{f_0} = \{-2 (z u_0 + z u_0^2) + z \mid z \in Z_{e_0}\}$$

(cf. [1]). Especially $U_{e_0} = U_{f_0}$ whenever $f = e + u_0$ with $u_0 \in U_{e_0}$.

Denote by $V_e$ the subspace $U_{e_0}$ of $Z_e$ spanned by all products $u u'$ for $u$ and $u' \in U_{e_0}$. We point out the relation between $V_{e_0}$ and $V_{f_0}$, where $f = e + u_0 + u_2$ with $u_0 \in U_{e_0}$ (we omit its verification), that is

$$(2.5) \quad V_{f_0} = \{-2 (u u_0 + u u_0^2) + v \mid v \in V_{e_0}\}.$$
In order to show that the above relation \( \rho \) is an equivalence relation, we shall make preparations beforehand.

**Lemma 4.** Let \( e, e' \) be idempotents of \( A \). If there exist \( w \in V_e^2 \) and \( w' \in V_{e'}^2 \) such that \( e + w = e' + w' \). Then \( V_e = V_{e'} \).

**Proof.** We can write \( e' = e + u + u^2 \) with some \( u \in U_e \) (cf. Observation (a)). Then \( w' = (w - u) + u^2 \) and \( w' u - u^2 = -u^3 (u^3 = 0 \) from (1.4)). Thus \( e' = e + u \) and \( u = w - w' \in U_0 \). Then we obtain \( U_{e'} = U_e \) as already stated (cf. Observation (d)). Now \( V_e = V_{e'} \) is clear. \( \square \)

**Theorem 5.** Let \( e, e' \) be idempotents of \( A \). Then there holds \( epe' \) if and only if \( V_e = V_{e'} \) and there exists \( w \in V_e^2 \) such that \( e + w = e' + w' \).

**Proof.** Suppose that there holds \( epe' \). Then by the definition of \( \rho \), there exists an idempotent \( f \in I(A) \) and \( w, w' \in V_f^2 \) such that \( e = f + w \) and \( e' = f + w' \). In this case, by means of Lemma 4, we have \( V_e = V_f = V_{e'} \) and \( e' = e + (w' - w) \) with \( w' - w \in V_f^2 = V_e^2 \). The converse is obvious by the definition of \( \rho \). \( \square \)

**Theorem 6.** The above defined relation \( \rho \) is an equivalence relation in \( I(A) \).

**Proof.** This relation is clearly reflexive and symmetric. Now it is nothing but a consequence of Theorem 5 that \( \rho \) is transitive. \( \square \)

The equivalence relation, denoted by \( R \), introduced by S. González and C. Martínez[1], means that, for idempotents \( ef \in I(A) \), \( eRf \) if and only if \( f = e + u \) with some \( u \in U_0 \) (cf. [1]). Notice that the equivalence relation \( \rho \) is more strict than \( R \) in the sense that \( e\rho f \) always implies \( eRf \).

**Theorem 7.** Each equivalence class of idempotents by \( \rho \) consists of only one element if and only if \( V_e^2 = 0 \) for each idempotent \( e \).

**Proof.** Suppose that each equivalence class consists of only one element; that is, \( e' \rho e \) for each idempotents \( e, e' \in I(A) \) implies \( e = e' \). In this case, if \( w \in V_e^2 \), then \( epe' + w \). By the hypothesis \( e = e + w \). So \( w = 0 \). This means that \( V_e^2 = 0 \). The converse is obvious. \( \square \)

P. Holgate introduced the notion of an orthogonal Bernstein algebra, which seems important to us to classify all Bernstein algebras: A Bernstein algebra \( A \) is said to be orthogonal if and only if \( U_e V_e = 0 \) for some idempotent \( e \) of \( A \) (cf. [2]). Then, as a corollary to Theorem 7, we shall obtain the following theorem.

**Theorem 8.** Let \( A \) be a Bernstein algebra and \( e \in I(A) \). Suppose that every equivalence class in \( I(A) \) by the relation \( \rho \) consists of only one element. Then \( U_e V_e = 0 \), that is, \( A \) is orthogonal, if and only if \( V_e = V_f \) for all \( f \in I(A) \).

**Proof.** Suppose \( U_e V_e = 0 \). Let \( f \) be an arbitrary idempotent of \( A \). Then there exists \( u \in U_e \) such that \( f = e + u + u^2 \). By the hypothesis and Theorem 7, we have \( uv = 0 \) and \( u^2 v = 0 \) for all \( v \in V_e \). So \( (2.5) \) \( V_f = (-2vu + vu^2) + v \mid v \in V_e \) = \( V_e \).

Reciprocally, suppose that \( V_e = V_f \) for all \( f \) in \( I(A) \). Then, for an arbitrary \( u \in U_e \), we have \( V_f = (-2uv + v \mid v \in V_e) \) with \( f = e + u + u^2 \) in view of \( (2.5) \) and \( vu^2 \in V_e^2 = 0 \). Thus the relation \( V_e = V_f \) means that \( uv = 0 \) for all \( u \in U_e \) and \( v \in V_e \). This implies \( U_e V_e = 0 \). \( \square \)

**References**

