On Bernstein Algebras with Low-dimension Subspaces $U^2$

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Abstract

The aim of this paper is to study some relations between dimensions of the subspaces $UZ$, $Z^2$, $U^2$ and $U^3$ of a finite-dimensional Bernstein algebra $A = Ke + U + Z$. The main results are concerned in the dependency of $\dim U^3$ on $\dim U^2$ in the case of $\dim U^2 \leq 4$.

§ 0. Introduction

A nonassociative, commutative algebra $A$ over a field $K$ is called a Bernstein algebra if there exists a nonzero algebra homomorphism $\omega: A \to K$ that satisfies

$$(x^2)^2 = \omega(x)^2 x^2$$

for every $x \in A$. We suppose that $K$ is an infinite field of characteristic different from 2 and that the dimension of a vector space $A$ over $K$ is finite.

It is known that a Bernstein algebra has always idempotents. If $e$ is an idempotent of $A$, then $A$ has a Peirce decomposition $A = Ke + U + Z$, where $Ker(\omega) = U + Z$, $U = \{x \in A | \omega(x) = 0\}$, $Z = \{x \in A | \omega(x) = 0\}$.

It is well known that the subspaces $U$ and $Z$ satisfy the following:

(a) $U \subseteq Z$, (b) $UZ + Z^2 \subseteq U$, (c) $UZ = \{0\}$.

It is also well known that the following identities hold for $u, u_1, u_2, u_3, u_4 \in U$ and for $z, z_1, z_2 \in Z$:

(d) $u^3 = 0, u_1^3 u_2 = -2u_1 (u_3 u_2)$,

(e) $(u^2)^2 = 0, u_1^2 u_2 u_3 = -2(u_1 u_3) u_2$,

(f) $u(uz) = 0, u_1(u_2 z) + u_2(u_1 z) = 0$;

(g) $(uz)^2 = 0, (u_1 z)(u_2 z) = 0$,

$(u_1 z)(u_2 z) = 0$,

$(u_1 z_1)(u_2 z_2) = -(u_1 z_2)(u_2 z_1)$.

On the other hand, it is known that the set of idempotents of $A$ is given by $\{e + u + u^2 | u \in U\}$ and that if $A = Ke + U + Z$ is a Peirce decomposition of $A$ with respect to another idempotent $\tilde{e} := e + \tilde{u} + \tilde{u}^2$, then

(h) $U = \{u + 2u \tilde{u} | u \in U\}$ and

$Z = \{z - 2(u \tilde{u}^2) z | z \in Z\}$.

Moreover, it is known that, $\dim U$ (and so also $\dim Z$) is an invariant of $A$, that is, it does not depend on the choice of the particular idempotent. Furthermore $\dim U^2$ and $\dim (UZ + Z^2)$ are also invariants of $A$. These invariants play a fundamental role in the problem of classifying all finite-dimensional Bernstein algebras, that is yet to be solved. In connection with that we are interested in possible combinations of the values of $\dim U^2, \dim U^3, \dim UZ, \dim Z^2$.

In the following the subspace spanned by $x_1, x_2, \ldots, x_i \in A$ over $K$ is denoted by $\langle x_1, \ldots, x_i \rangle_K$, or simply $\langle x_1, \ldots, x_i \rangle$ if there exist no apprehensions of misinterpretations.

§ 1. Sufficient conditions for $\dim U^2 \leq 1$

We state some sufficient conditions for $\dim U^2$ to be equal to or less than 1.
Proposition 1. If \( \dim Z^2 = \dim U \), then \( U^2 = \langle 0 \rangle \).

\[ \text{proof. Since } U = Z^2 \text{ by assumption and (a), it is clear that } U^2 = \langle 0 \rangle \text{ from (c).} \] \]

Proposition 2. If \( \dim Z^2 = \dim U - 1 \), then \( U^2 \leq 1 \).

\[ \text{proof. Assume that } \dim Z^2 = \dim U - 1 \text{. Then there exists } u \neq 0 \in U \text{ such that } U = Z^2 + Ku. \text{ Hence } U^2 = UZ^2 + uU \subset u(Z^2 + Ku) \subset Ku^2 \text{ by (c).} \]
\[ \therefore \dim U^2 \leq \dim K u^2 = 1. \] \]

Proposition 3. If there exists a nonzero element \( z \) in \( Z \) such that \( U = Uz \), then \( U^2 = \langle 0 \rangle \).

\[ \text{proof. By assumption there exist a basis } \{ u_i (1 \leq i \leq p) \} \text{ of } U \text{ and an element } z \neq 0 \text{ of } Z \text{ such that } \{ u_i z (1 \leq i \leq p) \} \text{ is a basis of } U. \text{ Then, } u_i = \sum_j \alpha_j u_j \text{ with } \alpha_j \in K \text{ for each } i. \]
\[ 1 \leq i \leq p, \text{ we have } u_i u_k = \sum \alpha_j \alpha_k (u_j) (u_k) = 0 \text{ from (g).} \] \]

Proposition 4. If \( \dim U = \dim UZ = 1 \), then \( U^2 = \langle 0 \rangle \).

\[ \text{proof. Let } z_1, \ldots, z_q \text{ be a basis of } Z \text{ and } U = Ku, u \neq 0. \text{ Since } UZ = U, \text{ there exist } \alpha_1, \ldots, \alpha_q \text{ in } K \text{ such that } u z_i = \alpha_i u \text{ (1 \leq i \leq q)}, \text{ where at least one element, e.g. } \alpha_1, \text{ is not 0. Then } u^2 = \alpha_1^{-2} (u z_i)^2 = 0 \text{ by (g).} \] \]

\section{The case \( U^2 = \langle 0 \rangle \) with \( UZ = U \)}

Proposition 5. If \( \dim UZ = \dim U, U^2 = \langle 0 \rangle \), and \( \dim Z = 1 \), then it is reduced to the case \( Z^2 = \langle 0 \rangle \).

\[ \text{proof. The condition that } \dim UZ = \dim U \text{ is equivalent to } UZ = U \text{ by (b). Thus we show that, if } Z^2 = \langle 0 \rangle \text{, one can choose proper idempotent } \bar{e} \text{ so that } A = K \bar{e} + U + Z \text{ satisfying that } C \bar{e} = U, \bar{e}^2 = \langle 0 \rangle, \text{ and } Z^2 = \langle 0 \rangle. \]

Now choose one nonzero element \( z_1 \) of \( Z \) and put \( u_1 = z_1^2 \). If \( p = \dim U \), then \( Z^2 = K z_1^2 \) and there exists a basis \( \{ u_1, u_2, \ldots, u_p \} \) of \( U \) with \( u_1 = z_1^2 \). Since \( UZ = U, u_1 z_1, \ldots, u_p z_1 \text{ are linearly independent. Hence, if we write } u_i z_1 = \sum_j \alpha_j u_j \]
\[ (i = 1, \ldots, p), \text{ then the determinant } \Delta = \det [\alpha_j] \text{ is not 0. Then, the linear equation } \sum \lambda_i (u z_i) z_1 = 0 \text{ with } \lambda_i \in K \text{ implies that } \sum \lambda_i (\sum_j \alpha_j u_j u z_i) = \sum \lambda_i (\sum_j \alpha_j a_j u z_i) = 0, \text{ therefore, } \sum_j \alpha_j a_j = 0 \text{ for each } j \text{ and the determinant of this linear system of linear equations is identical to } \Delta. \text{ Because } \Delta \neq 0, \text{ we have } \lambda_i = 0 \text{ for all } i, \text{ which means that } (u z_i) z_1, \ldots, (u p z_i) z_1 \text{ are linearly independent and there exist uniquely } \beta_i (i = 1, \ldots, p) \text{ in } K \text{ such that } u z_i = \sum \beta_i (u z_i) z_1. \text{ Now, if we define } u := \frac{1}{4} \sum \beta_i u_i \text{ and } \bar{e} := e + u, \text{ then, from (h), we get } U = \{ u + 2 u a | u \in U \} = U \text{ and } Z = \{ z - 2 a z | z \in Z \} = K (z_1 - 2 a z_1) \text{ with } (z_1 - 2 a z_1)^2 = 0 \text{ by } U^2 = \langle 0 \rangle \text{ and (g).} \]

\section{Some consequences of \( \dim U^2 = 1 \)}

Theorem 1. If \( \dim U^2 = 1 \), then \( U^2 = \langle 0 \rangle \) and \( (U^2)^2 = \langle 0 \rangle \).

\[ \text{proof. Let } \{ u_i | i = 1, \ldots, p \} \text{ be a basis of } U. \text{ Then, by assumption, there exists at least one nonzero element in the set } \{ u_i u_j | 1 \leq i, j \leq p \}. \text{ Let } z_i := u_i u_j u_0 \neq 0 \text{ and put } u_i u_j = a_i z_j \text{ with } a_i \in K \text{ for each pair } i, j (1 \leq i, j \leq p). \text{ Then there occur two possible cases: } i_0 = j_0 \text{ or } i_0 < j_0. \text{ We prove the assertion in each case.} \]

1) If \( i_0 = j_0 \), then we can assume that \( i_0 = j_0 = 1 \), i.e., \( z_1 = u_1 u_2 \) without loss of generality. From (d) we have that \( z_1 u_i = -2 a_i u_1 u_2 = 0 \) for all \( j \), which means that \( U^2 = \langle 0 \rangle \). On the other hand, since \( U^2 = K z_1^2 \text{ and } z_2^2 = 0 \text{ from (d), we get also } (U^2)^2 = \langle 0 \rangle. \]

2) If \( i_0 < j_0 \), then we can put \( (i_0, j_0) = (1, 2) \), i.e., \( z_1 = u_1 u_2 \) without loss of generality. Then, by assumption and (d), it holds that

\[ (1) \quad (u_1^2)^2 = a_1 u_2^2 = 0, \quad (u_2^2)^2 = a_2 u_1^2 = 0; \]

\[ (2) \quad z_1 u_i = -\frac{1}{2} u_1^2 u_i u_2 = -\frac{1}{4} a_1 z_1 u_i u_2^2 = \frac{1}{4} a_{12} z_1 u_i; \]

and in like manner

\[ (3) \quad z_2 u_1 = -\frac{1}{4} a_{12} z_1 u_2; \]

\[ (4) \quad z_i u_i = -\frac{1}{2} a_1 z_2 u_1 + \frac{1}{2} a_2 z_1 u_2 \]

for all \( i (3 \leq i \leq p) \).
The equation (1) implies the following
\[(5) \; x_1^2 = 0 \quad \text{or} \quad a_{11} = a_{22} = 0.\]
Thus, if \(a_{11}a_{22} \neq 0\), then \(U^0 = \langle 0 \rangle\) by virtue of (2), (3) and (4), and moreover \(z_i^2 = -\frac{1}{2} u_i u_j^2 \) = \(-\frac{1}{2} a_{ij} u_i u_j^2\) by (5). On the contrary, if \(a_{11} a_{22} = 0\), then this case belongs to the case \(i_0 = j_0 = 1\), since \(u_i^2 = a_{ii} z_i \neq 0\).

Corollary 1. If \(\dim U^2 = \dim Z = 1\), then \(UZ + Z^2 = \langle 0 \rangle\).

proof. The claim follows from Theorem 1 since \(U^2 = Z\).

Theorem 2. If \(\dim U^2 = 1\) and \(\dim Z = 2\), then \(\dim UZ < \dim U\).

proof. Let \(\{u_1, \ldots, u_p\}\) be a basis of \(U\) and choose a basis \(\{z_1, z_2\}\) of \(Z\) such that \(U^2 = Kz_2\). Then we get \(UZ = \langle u_1 z_1, u_2 z_2 \rangle\) as a corollary of Theorem 1. Therefore \(UZ = \langle u_1 z_1, u_2 z_2 \rangle\). Put \(u_i z_1 = a_{i1} z_1 + a_{i2} z_2\) for every \(i, j\) and (f) \(\alpha_j \neq 0\). Then, since \(u_1 (u_2 z_1) + u_2 (u_2 z_2) = 0\) and \((u_2 z_2)(u_2 z_2) = 0\) from (f) and (g), respectively, the following equations hold for each pair \(i, j\):
\[(1) \quad \sum_{k} \alpha_{jk} a_{ik} = \sum_{k} \alpha_{jk} a_{ik} = 0,\]
\[(2) \quad \sum_{k} \alpha_{jk} a_{ik} = 0.\]

Define \(i_k = \sum_{k} \alpha_{jk} a_{ik} = \sum_{k} \alpha_{jk} a_{ik}\) and \(a_{ik} = \sum_{k} \alpha_{jk} a_{ik}\) for each \(k\). Then, from (1) \(0 = \sum_{k} \sum_{j} \alpha_{jk} a_{ik} = \sum_{k} \sum_{j} \alpha_{jk} a_{ik}\)\(\sum_{k} \alpha_{jk} a_{ik} = 0\). Therefore
\[(3) \quad \sum_{k} \alpha_{jk} a_{ik} = 0 \quad \text{for} \quad i = 1, \ldots, p.\]

There are two possible cases.

If \(i_k = 0\) for all \(k\), then, putting \(u_0 = \sum_{i} u_i\), we have that \(u_0 z_1 = \sum_{i} u_i z_1 = \sum_{i} \sum_{k} \alpha_{jk} a_{ik} u_i\) = \(\sum_{i} \sum_{k} \alpha_{jk} a_{ik} u_i\) = \(0\). Therefore, by adopting \(\{u_0, u_1, \ldots, u_p\}\) as a basis of \(U\), we get that \(U^2 = K(u_0 z_1)\). So \(\dim UZ = 2 < \dim U\).

If \(i_k \neq 0\) for some \(k_0\), then there exist two situations:

i) If \(\sum_{k} \alpha_{jk} a_{ik} \neq 0\) for some \(i\), then \(\xi_{1}, \ldots, \xi_{p}\) defined by \(\xi_{i} = \sum_{k} \alpha_{jk} a_{ik} (i = 1, \ldots, p)\) satisfy the simultaneous equations \(\sum_{k} \alpha_{jk} x_{k} = 0\) for \(j = 1, \ldots, p\), which is shown from (2), and \(\xi_{i} \neq 0\) for some \(i\) by assumption. Therefore \(\det(i_{jk})_{k=1} = 0\), which means that \(u_1 z_2, \ldots, u_p z_2\) are linearly dependent and \(\dim UZ < 2\).

ii) If \(\sum_{k} \alpha_{jk} x_{k} = 0\) for all \(i\), then equations (3) is reduced to
\[(4) \quad \sum_{k} \alpha_{jk} x_{k} = 0 \quad \text{for} \quad i = 1, \ldots, p.\]
Since \(u_i \sum_{k} \alpha_{jk} x_{k} = \sum_{k} \alpha_{jk} x_{k} = \alpha_{i2} z_1\) for all \(i\), the claim that \(a_{i2} = 0\) for all \(k\) is contrary to the assumption that \(\dim U^2 = 1\) and we can conclude that \(a_{i2} \neq 0\) for some \(k\). Then the simultaneous equations \(\sum_{k} x_{k} = 0\) for \(i = 1, \ldots, p\) have non-trivial solutions \(x_{k} = \alpha_{i2} z_1 = 1, \ldots, p\), which means that \(\dim UZ < 2\).

§ 4. The case \(\dim U^2 = 2, 3,\) or 4

First of all we state two lemmas which will be used in the proofs of the theorems following below. The first lemma is elementary.

Lemma 1. Let \(B = \{a_1, a_2, \ldots, a_k\} (k > 0)\) be a basis of a \(k\)-dimensional vector space \(V\) and \(b = \lambda a_i\) a nonzero vector with some \(\lambda_i \neq 0\). Then, also the set \(\{a_1, a_2, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k\}\) obtained from \(B\) by replacing \(a_i\) with \(b\) is a basis of \(V\).

Lemma 2. Let \(\{u_1, \ldots, u_p\}\) be a basis of \(U\) and \(\{z_r = u_{i,j} | r = 1, \ldots, k\}\) a basis of \(U^2\), where we suppose that \(k = \dim U^2\), \(1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq p\), \(1 \leq j_1 \leq j_2 \leq \ldots \leq j_k \leq p\), and \(i_1, j_1, \ldots, i_k, j_k\), for \(r = 1, 2, \ldots, k\). If \(X\) is a subspace of \(U\) and \(z_{r-j} \in X\) for each \(r = 1, \ldots, k\) and each \(t (1 \leq t \leq p)\), then \(z_{r-j} \in X\) for each \(r = 1, \ldots, k\) and each \(t (1 \leq t \leq p)\).

proof. We shall put \(z_r = u_{i,j} (i = i_r, j = j_r)\) for any \(s (1 \leq s \leq k)\) and \(u_{i,j} = \Sigma_{r} z_r, u_{i,j} = \Sigma_{r} z_r\) for each \(t = 1, \ldots, p\) with \(t = (i_1, i_2, j_2, \ldots, i_k, j_k)\). Then, by (d), \(z_{r-j} = - (u_{i,j} - (u_{i,j}) u_i = - \Sigma_{r} z_r - \Sigma_{r} z_r u_i\), where \(z_{r-j} \in X\) by assumption. Therefore \(z_{r-j} \in X\).

Theorem 3. If \(\dim U^2 = 2\), then \(U^2 = \langle 0 \rangle \) or else \(U^2 = \langle U^2 \rangle \) and, in either case, \(\dim U^2 \leq 1\).

proof. Let \(u_1, \ldots, u_p (p \geq 2)\) be a basis of \(U\). Then, by assumption, we can choose two
products $u_iu_j$ and $u_ku_l$ with $i\leq j, k\leq l, i \leq k,$ $(i, j) \neq (k, l)$, as a basis of $U^2$. On the other hand, we can easily see that each possible combination of $u_i, u_j; u_k, u_l$, denoted simply $[i, j], [k, l]$, belongs to one of the following five types, by changing the number $i$ of $u_i$, if necessary:

1: $[11, 22]$ (i.e. $u_1^2, u_2^2)$
2: $[11, 12]$ (i.e. $u_1^2, u_{12}$)
3: $[11, 23]$ (i.e. $u_1^2, u_{13}$)
4: $[12, 13]$ (i.e. $u_{12}, u_{13}$)
5: $[12, 34]$ (i.e. $u_{12}, u_{34}$)

Once a basis $z_1, z_2$ of $U^2$ is chosen, each remaining product $u_iu_j \in U^2$ will be written $u_iu_j = \alpha_{ij}z_1 + \beta_{ij}z_2$ with $\alpha_{ij}, \beta_{ij} \in K$. We shall establish the assertion of the theorem for each type, separately.

**Type 1: [11, 22]**
We put $z_1 = u_1^2$ and $z_2 = u_2^2$. Then by (d)

1. $z_1u_1 = z_2u_2 = 0$, $z_1u_2 = 4\alpha_{12}\beta_{12}z_1u_2$.
and by (e)

2. $z_1^2 = z_2^2 = 0$, therefore

$z_1z_2 = -2(u_1u_2)^2 = -4\alpha_{12}\beta_{12}z_1z_2$.

If $4\alpha_{12}\beta_{12} = -1$, then we can conclude from (1) that $z_iu_j = 0$ for $i, j = 1, 2$, which implies by virtue of Lemma 2 that $z_iu_i = z_2u_i = 0$ for all $i (1 \leq i \leq p)$. Consequently we have $U^2 = 0$. On the contrary, if $4\alpha_{12}\beta_{12} \neq \pm 1$, we obtain that $U^2 \subset K_1u_2$ and $\dim U^2 \leq 1$ from (1). Also $(U^2)^2 = 0$ from (2). □

**Type 2: [11, 12]**
We put $z_1 = u_1^2$ and $z_2 = u_{12}u_2$. If $\beta_{22} \neq 0$, then this type is reduced to Type 1, as is seen easily by Lemma 1. Therefore we can suppose $\beta_{22} = 0$, that is, $u_2^2 = \alpha_{22}z_1$. Define $X$ by $X := Kz_1u_2$. Then by (d) and by assumption

1. $z_1u_1 = 0$, $z_2u_1 = -\frac{1}{2}z_2u_2 \in X$, so

$z_2u_2 = -\frac{1}{2}z_2u_2z_1u_1 = 0$.

therefore by Lemma 2

2. $z_iu_i \in X$, $z_iu_1 \in X$ for all $i (1 \leq i \leq p)$.

Consequently we have $U^2 \subset X$ and so forth. Moreover, (e) and $(u_1u_2)^2 = -u_1^2u_2^2 = -\frac{1}{2}z_2u_2^2 = 0$ imply that $(U^2)^2 = 0$.

**Type 3: [11, 23]**
We put $z_1 = u_1^2$, and $z_2 = u_{13}$. If there exists at least one nonzero element in $(\beta_{22}, \beta_{33})$ or $(\alpha_{11}, \alpha_{33}, \beta_{12}, \beta_{13})$, then this type is reduced to Type 1 or Type 2, respectively, as is shown by Lemma 1. Therefore we can suppose that $\alpha_{22} = \beta_{33} = \beta_{12} = \beta_{13} = 0$, i.e., $u_2^2 = u_3^2 = 0$, $u_{12} = \alpha_{12}u_2$ and $u_{13} = \alpha_{13}u_3$. Then by assumption and (d)

1. $z_iu_i = z_4u_i = 0$ for $i = 1, 2, 3$.

Consequently by Lemma 2

2. $z_iu_i = z_3u_i = 0$ for all $i (1 \leq i \leq p)$, which means that $U^2 = \langle 0 \rangle$.

**Type 4: [12, 13]**
We put $z_1 = u_{12}u_2$ and $z_2 = u_{13}$. If there exists at least one nonzero element in $(\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{23})$ or $(\beta_{22}, \alpha_{33}, \beta_{12}, \beta_{13})$, then this type is reduced to Type 2 or Type 3, respectively, as is shown by Lemma 1. Hence we can suppose that $\alpha_{22} = \beta_{33} = \beta_{12} = \beta_{13} = 0$, i.e., $u_2^2 = u_3^2 = 0$ for $i = 1, 2, 3$ and $u_{12}u_3 = \alpha_{23}z_1 + \beta_{23}z_2$. Now define the subspace $X$ of $U$ by $X := Kz_1u_3$. Then by assumption and (d)

1. $z_1u_1 = z_1u_2 = z_2u_1 = z_2u_3 = 0$, $z_2u_2 = -z_1u_3$.

Therefore by Lemma 2

2. $z_iu_i \in X$, $z_iu_1 \in X$ for $i (1 \leq i \leq p)$, which means that $U^2 \subset X$ and so forth. On the other hand, by assumption and (e), $z_i^2 = z_i^2 = z_1z_2 = 0$. This implies that $(U^2)^2 = \langle 0 \rangle$.

**Type 5: [12, 34]**
We put $z_1 = u_{12}u_2$ and $z_2 = u_{13}u_4$. If there exists at least one nonzero element in $(\alpha_{11}, \alpha_{22}, \alpha_{33}, \alpha_{44})$, $(\beta_{11}, \beta_{22}, \beta_{33}, \beta_{44})$, or $(\alpha_{12}, \alpha_{23}, \beta_{13}, \beta_{23}, \alpha_{14}, \alpha_{24}, \beta_{14}, \beta_{24})$, then this type is reduced to Type 2, 3, or 4, respectively, as is shown by Lemma 1. Therefore we can assume that $u_{12}u_i = 0$ for all $(i, j) \neq (1, 2), (3, 4), (1 \leq i, j \leq 4)$. Then by assumption and (d)

1. $z_1u_1 = z_1u_2 = z_2u_3 = z_2u_4 = 0$,

$z_1u_3 = z_1u_4 = z_2u_1 = z_2u_2 = 0$.

Then by Lemma 2

2. $z_iu_i = z_3u_i = 0$ for all $i (1 \leq i \leq p)$.

Therefore $U^2 = \langle 0 \rangle$ and so forth. □

**Corollary 2.** If $\dim U^2 = 2$ and $U^2 = Z$, then $UZ = \langle 0 \rangle$ or else $Z^2 = \langle 0 \rangle$, and, in either case, $\dim UZ \leq 1$. 

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Theorem 4. If $\dim U^2 = 3$, then $\dim U^0 \leq 2$ or else $(U^0)^2 = \langle 0 \rangle$ and, in either case, $\dim U^0 \leq 3$.

proof. We shall prove the theorem in the same method as one for Theorem 3 that is composed of classifying the types of base elements of $U^0$ and computing separately according to the types, reducing to the established types before by virtue of Lemma 1 and Lemma 2. However, in order to avoid redundancy, we shall describe in the following only results of verification omitting the detail of computing.

Let $u_1, \ldots, u_p$ be a basis of $U$. Then, by assumption, we can choose three products $u_{ij} = u_i u_j$ and $u_{mn} = u_m u_n$ with $i \neq j, k \neq l, m \neq n$, $i \leq k \leq m$, $(i, j) \neq (k, l) \neq (m, n)$, as a basis of $U^0$. Then each possible combination in three products, denoted by $[ij, kl, mn]$ for short, belongs to one of the fourteen types listed below by changing the number $i$ of $u_i$, if necessary:

13: [12, 13, 45]  14: [12, 34, 56]

Once a basis $z_1, z_2, z_3$ of $U^2$ are chosen, every remaining product $u_{ij} \in U^2$ will be written $u_{ij} = \alpha_{ij} z_1 + \beta_{ij} z_2 + \gamma_{ij} z_3$ with $\alpha_{ij}, \beta_{ij}, \gamma_{ij} \in K$.

Type 1: [11, 12, 22]

We put $z_1 = u_1^2, z_2 = u_1 u_2$, and $z_3 = u_2^2$. Then it is shown that $U^0 \subset \langle z_1 u_2, z_3 u_1 \rangle$ and $\dim U^0 \leq 2$.

Type 2: [12, 22, 23]

We put $z_1 = u_2^2, z_2 = u_1 u_2$, and $z_3 = u_3 u_3$. Omitting the case that is reduced to Type 1, we can assume that $\alpha_{33} = \gamma_{13} = 0$. Then it is shown that $U^0 \subset \langle z_1 u_2, z_1 u_3, z_3 u_1 \rangle$ and that $z_1 u_2, z_1 u_3, z_3 u_1$ are linearly dependent, so $\dim U^0 \leq 2$.

Type 3: [11, 12, 23]

We put $z_1 = u_1^2, z_2 = u_1 u_2$, and $z_3 = u_2 u_3$. Omitting the case that is reduced to Type 1 or Type 2, we can assume that $\beta_{12} = \beta_{33} = \gamma_{23} = 0$. Then it is shown that $U^0 \subset \langle z_1 u_2, z_3 u_1 \rangle$ and so forth.

Type 4: [11, 22, 33]

We put $z_1 = u_1^2, z_2 = u_2^2$, and $z_3 = u_3^2$. Omitting the case that is reduced to Type 1 or Type 2, we can assume that $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0$ for all $i, j$ where $i \neq j \neq 3$. Then it is shown that $U^0 = \langle 0 \rangle$ and that, together with the preceding results, $\dim U^0 \leq 2$.

Type 5: [12, 22, 23]

We put $z_1 = u_1 u_2, z_2 = u_2^2$, and $z_3 = u_3^2$. Omitting the case that is reduced to Type 1, 2, or 3, we can assume that $\alpha_{11} = \gamma_{11} = \gamma_{33} = \beta_{11} = \beta_{33} = \alpha_{33} = \gamma_{33} = 0$. Then it is shown that $U^0 \subset \langle z_1 u_2, z_1 u_3, z_2 u_3 \rangle$ and that $(U^0)^2 = \langle 0 \rangle$.

Type 6: [12, 13, 23]

We put $z_1 = u_1 u_2, z_2 = u_2 u_3$, and $z_3 = u_3 u_3$. Omitting the case that is reduced to Type 2 or Type 5, we can assume that $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0$ for $i = 1, 2, 3$. Then it is shown that $U^0 \subset \langle z_1 u_3, z_1 u_1 \rangle$ and so forth.

Type 7: [11, 22, 34]

We put $z_1 = u_1^2, z_2 = u_2^2$, and $z_3 = u_3 u_4$. Omitting the case that is reduced to Type 1, 2, 3, or 4, we can assume that $\alpha_{13} = \beta_{13} = \gamma_{13} = \alpha_{44} = \beta_{44} = \gamma_{44} = \gamma_{12} = \beta_{12} = \gamma_{12} = \beta_{12} = \gamma_{12} = \gamma_{24}$ = 0. Then it is shown that $U^0 \subset Kz_2 u_2$ and so forth.

Type 8: [11, 12, 34]

We put $z_1 = u_1^2, z_2 = u_1 u_2$, and $z_3 = u_3 u_4$. Omitting the case that is reduced to Type 1, 2, 3, or 7, we can assume that $\beta_{33} = \gamma_{33} = \beta_{44} = \gamma_{44} = \gamma_{12} = \beta_{12} = \gamma_{12} = \beta_{12} = \gamma_{12} = \gamma_{24}$ = 0. Then it is shown that $U^0 \subset Kz_2 u_2$ and so forth.

Type 9: [11, 23, 24]

We put $z_1 = u_1^2, z_2 = u_2 u_3$, and $z_3 = u_3 u_4$. Omitting the case that is reduced to Type 2, 3, 5, 7, or 8, we can assume that $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0$ for $j = 2, 3, 4$. Then it is shown that $U^0 \subset Kz_2 u_4$ and so forth.

Type 10: [12, 13, 14]

We put $z_1 = u_1 u_2, z_2 = u_1 u_3$, and $z_3 = u_1 u_4$. Omitting the case that is reduced to Type 3, 5, 6, or 9, we can assume that $\alpha_{ii} = \beta_{ii} = \gamma_{ii} = 0$ for $i = 1, 2, 3, 4$ and $\alpha_{34} = \beta_{24} = \gamma_{23} = 0$. Then it
is shown that \( U^0 \subseteq \langle z_1 u_2, z_2 u_3, z_3 u_4 \rangle \) and that 
\((U^0)^2 = \langle 0 \rangle \). □

Type 11: [12, 13, 24]
We put \( z_1 = u_1 u_2, z_2 = u_1 u_3, \) and \( z_3 = u_2 u_4 \).
Omitting the case that is reduced to Type 3, 5, 6, 8, 9, or 10, we can assume that \( \alpha_i = \beta_2 = \gamma_2 = 0 \) for \( i = 1, 2, 3, 4 \) and \( \beta_4 = \gamma_4 = 0 \) for \( (i, j) = (1, 4), (2, 3) \). Then it is shown that 
\( U^0 \subseteq \langle z_1 u_2, z_2 u_3, z_3 u_4 \rangle \) and so forth. □

Type 12: [11, 23, 45]
We put \( z_1 = u_1^2, z_2 = u_2 u_3, \) and \( z_3 = u_4 u_5 \).
Omitting the case that is reduced to Type 2, 3, 7, 8, 9, or 11, we can assume that \( \alpha_i = \beta_2 = \gamma_2 = 0 \) for \( i = 2, 3, 4, 5 \) and \( \alpha_4 = \beta_4 = \gamma_4 = 0 \) for \( i = 2, 3 \) and \( j = 4, 5 \) and \( \beta_4 = \gamma_4 = 0 \) for \( i = 2, 3, 4, 5 \). Then it is shown that 
\( U^0 = \langle 0 \rangle \) and so forth. □

Type 13: [12, 13, 45]
We put \( z_1 = u_1 u_2, z_2 = u_1 u_3, \) and \( z_3 = u_3 u_5 \).
Omitting the case that is reduced to Type 3, 5, 6, 8, 9, 10, 11 or 12, we can assume that \( \alpha_i = \beta_2 = \gamma_2 = 0 \) for \( i = 1, 2, 3, 4, 5 \) and \( \alpha_4 = \alpha_{12} = \beta_4 = \beta_5 = \gamma_4 = \gamma_5 = \gamma_2 = \gamma_3 = \gamma_3 = 0 \). Then it is shown that 
\( U^0 \subseteq \langle z_1 u_2, z_2 u_3, z_3 u_5 \rangle \) and so forth. □

Type 14: [12, 34, 56]
We put \( z_1 = u_1 u_2, z_2 = u_3 u_4, \) and \( z_3 = u_3 u_6 \).
Omitting the case that is reduced to Type 8, 10, 11, 12, or 13, we can assume that \( u_m u_j = 0 \) for every pair \( (i, j) \neq (1, 2), (3, 4), (5, 6) (1 \leq i, j \leq 6) \). Then it is shown that \( U^0 = \langle 0 \rangle \) and so forth. □

Corollary 3. If \( \dim U^0 = 3 \) and \( U^0 = \mathbb{Z} \), then 
\( \dim UZ \leq 2 \) or else \( Z^0 = \langle 0 \rangle \), and, in either case, 
\( \dim UZ \leq 3 \).

Theorem 5. If \( \dim U^0 = 4 \), then \( \dim U^0 \leq 5 \) or else \( (U^0)^2 = \langle 0 \rangle \) and, in either case, 
\( \dim U^0 \leq 6 \).

proof. We shall state here only results of verification omitting the detail of computing.

Let \( u_1, \ldots, u_p \) be a basis of \( U \). Then, by assumption, we can choose four products \( z_1 = u_1 u_5, z_2 = u_2 u_1, z_3 = u_3 u_5, \) and \( z_4 = u_3 u_1 \), where

\[
i \leq j, k \leq l, m \leq n, s \leq t, i \leq k \leq m \leq s, (i, j) \neq (k, l) \neq (m, n) \neq (s, t),\]

as a basis of \( U^0 \). Then each possible combination in the four products, denoted by \([ij, kl, mn, st]\), belongs to one of the following thirty-nine types by changing the number \( i \) of \( u_i \) if necessary:

<table>
<thead>
<tr>
<th>Type</th>
<th>Formulas</th>
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<tbody>
<tr>
<td>1</td>
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<td>2</td>
<td>([11, 12, 13, 23])</td>
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<tr>
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</table>

Type 1: [11, 12, 13, 22]
We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_1 u_3, \) and \( z_4 = u_2^3 \). Then it is shown that 
\( U^0 \subseteq \langle z_1 u_2, z_1 u_3, z_2 u_3, z_4 u_1 \rangle \) and so \( \dim U^0 \leq 4 \). □

Type 2: [11, 12, 13, 23]
We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_1 u_3, \) and \( z_4 = u_2 u_3 \). Omitting the case that is reduced to Type 1, we obtain that 
\( U^0 \subseteq \langle z_1 u_2, z_1 u_3, z_3 u_3, z_4 u_1 \rangle \) and so \( \dim U^0 \leq 4 \). □

Type 3: [11, 13, 22, 23]
We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_2^2, \) and \( z_4 = u_2 u_3 \). Omitting the case that is reduced to Type 1 or Type 2, we obtain that 
\( U^0 \subseteq \langle z_1 u_2, z_2 u_3, z_4 u_1 \rangle \) and that, together with the preceding results, 
\( \dim U^0 \leq 4 \). □

Type 4: [11, 12, 22, 33]
We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_2^2, \) and \( z_4 = u_2^3 \). Omitting the case that is reduced to Type 1?
or Type 3, we obtain that \( U^0 \subset \langle z_1 u_2, z_3 u_4 \rangle \) and that, together with the preceding results, \( \dim U^0 \leq 5 \).

**Type 5:** [11, 22, 33, 44]

We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_1 u_4, \) and \( z_4 = u_2^2 \). Omitting the case that is reduced to Type 4, we obtain that \( U^0 \subset \langle z_1 u_2, z_1 u_3, z_1 u_4, z_2 u_3, z_2 u_4, z_3 u_4, z_4 u_4 \rangle \) and that \( z_1 u_2, z_1 u_3, z_1 u_4, z_2 u_3, z_2 u_4, z_3 u_4, z_4 u_4 \) are linearly dependent, so \( \dim U^0 \leq 5 \), or else \( (U^0)^2 = \langle 0 \rangle \).

**Type 6:** [11, 13, 14, 22]

We put \( z_1 = u_1^2, z_2 = u_1 u_3, z_3 = u_1 u_4, \) and \( z_4 = u_2^2 \). Omitting the case that is reduced to Type 1, \( 2, 3, 4, \) or 5, we obtain that \( U^0 \subset \langle z_1 u_2, z_1 u_3, z_1 u_4, z_2 u_3, z_2 u_4, z_3 u_4, z_4 u_4 \rangle \) and that \( z_1 u_2, z_1 u_3, z_1 u_4, z_2 u_3, z_2 u_4, z_3 u_4, z_4 u_4 \) are linearly dependent, so \( \dim U^0 \leq 5 \), or else \( (U^0)^2 = \langle 0 \rangle \).

**Type 7:** [11, 12, 13, 14]

We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_1 u_4, \) and \( z_4 = u_2 u_4 \). Omitting the case that is reduced to Type 1, \( 2, \) or 6, we obtain that \( U^0 \subset \langle z_1 u_2, z_1 u_3, z_1 u_4, z_2 u_3, z_2 u_4, z_3 u_4, z_4 u_4 \rangle \) and that \( (U^0)^2 = \langle 0 \rangle \).

**Type 8:** [11, 14, 22, 33]

We put \( z_1 = u_1^2, z_2 = u_1 u_4, z_3 = u_2, \) and \( z_4 = u_1 u_4 \). Omitting the case that is reduced to Type 1, \( 3, 4, 5, \) or 6, we obtain that \( U^0 \subset \langle z_1 u_4, z_2 u_4 \rangle \) and so forth.

**Type 9:** [11, 12, 13, 24]

We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_4, \) and \( z_4 = u_4 u_4 \). Omitting the case that is reduced to Type 1, \( 2, 3, 4, 6, \) or 7, we obtain that \( U^0 \subset \langle z_1 u_2, z_2 u_3, z_2 u_4, z_4 u_4 \rangle \) and so on.

**Type 10:** [11, 12, 22, 34]

We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_1 u_4, \) and \( z_4 = u_3 u_4 \). Omitting the case that is reduced to Type 1, \( 8, 9, \) or 10, we obtain that \( U^0 \subset \langle z_1 u_2, z_3 u_4 \rangle \) and so forth.

**Type 11:** [11, 13, 22, 24]

We put \( z_1 = u_1^2, z_2 = u_1 u_3, z_3 = u_2, \) and \( z_4 = u_2 u_4 \). Omitting the case that is reduced to Type 1, \( 3, 4, 6, 9, \) or 10, we obtain that \( U^0 \subset \langle z_1 u_2, z_3 u_4 \rangle \) and so forth.

**Type 12:** [11, 12, 24, 33]

We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_2 u_4, \) and \( z_4 = u_2 \). Omitting the case that is reduced to Type 1, \( 2, 3, 4, 6, 8, 9, \) or 10, we obtain that \( U^0 \subset \langle z_1 u_2, z_3 u_4 \rangle \) and so forth.

**Type 13:** [11, 12, 23, 34]

We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_3 u_3, \) and \( z_4 = u_3 u_4 \). Omitting the case that is reduced to Type 1, \( 2, 3, 6, 8, \) or 9, 10, 11, or 12, we obtain that \( U^0 \subset \langle z_1 u_2, z_3 u_4, z_4 u_4 \rangle \) and so on.

**Type 14:** [11, 12, 23, 24]

We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_2 u_4, \) and \( z_4 = u_4 u_2 \). Omitting the case that is reduced to Type 1, \( 2, 3, 4, 6, 7, 8, 9, \) or 12, or 13, we obtain that \( U^0 \subset \langle z_1 u_2, z_3 u_1, z_4 u_1, z_4 u_4 \rangle \) and so on.

**Type 15:** [11, 23, 24, 34]

We put \( z_1 = u_1^2, z_2 = u_1 u_3, z_3 = u_4, \) and \( z_4 = u_4 u_4 \). Omitting the case that is reduced to Type 2, \( 6, 12, 13, \) or 14, we obtain that \( U^0 \subset \langle z_3 u_2, z_4 u_2 \rangle \) and so forth.

**Type 16:** [12, 13, 14, 23]

We put \( z_1 = u_1 u_2, z_2 = u_1 u_3, z_3 = u_1 u_4, \) and \( z_4 = u_4 u_4 \). Omitting the case that is reduced to Type 2, \( 7, 9, 13, 14, \) or 15, we obtain that \( U^0 \subset \langle z_2 u_2, z_3 u_3, z_3 u_4, z_4 u_4 \rangle \) and so forth.

**Type 17:** [12, 13, 24, 34]

We put \( z_1 = u_1 u_2, z_2 = u_1 u_3, z_3 = u_4, \) and \( z_4 = u_4 u_4 \). Omitting the case that is reduced to Type 9, \( 13, \) or 16, we obtain that \( U^0 \subset \langle z_1 u_4, z_2 u_2, z_3 u_1, z_4 u_2 \rangle \) and so forth.

**Type 18:** [11, 22, 33, 45]

We put \( z_1 = u_1^2, z_2 = u_2, z_3 = u_3^2, \) and \( z_4 = u_4 u_4 \). Omitting the case that is reduced to Type 2, \( 5, \) 8, 10, or 12, we obtain that \( U^0 \subset \langle z_1 u_2, z_2 u_3, z_4 u_3 \rangle \) and so forth.

**Type 19:** [11, 13, 22, 45]

We put \( z_1 = u_1^2, z_2 = u_1 u_3, z_3 = u_2, \) and \( z_4 = u_4 u_4 \). Omitting the case that is reduced to Type 1, \( 3, 4, 6, 9, \) or 10, we obtain that \( U^0 \subset \langle z_1 u_2, z_1 u_3, z_2 u_2 \rangle \) and so forth.

**Type 20:** [11, 22, 34, 35]

We put \( z_1 = u_1^2, z_2 = u_2, z_3 = u_3 u_4, \) and \( z_4 = u_3 u_3 \). Omitting the case that is reduced to Type 8, \( 9, \) 10, 12, 14, 15, 16, 18, or 19, we get that \( U^0 \subset \langle z_1 u_2, z_3 u_4 \rangle \) and so forth.

**Type 21:** [11, 12, 23, 45]

We put \( z_1 = u_1^2, z_2 = u_1 u_2, z_3 = u_2 u_3, \) and \( z_4 = u_4 u_4 \). Omitting the case that is reduced to Type
1 , 2 , 3 , 9 , 10 , 11 , 12 , 13 , or 19, we obtain that 
$U \subset \langle z_1 u_2 , z_1 u_3 \rangle$ and so forth. □

Type 22: [11, 12, 13, 45]
We put $z_1 = u_1^2$, $z_2 = u_2 u_3$, $z_3 = u_1 u_3$, and $z_4 = u_4 u_5$. Omitting the case that is reduced to Type 1 , 2 , 6 , 7 , 9 , 10 , 11 , 13 , or 21, we obtain that 
$U \subset \langle z_1 u_2 , z_1 u_3 , z_3 u_4 \rangle$ and so forth. □

Type 23: [11, 23, 24, 35]
We put $z_1 = u_1^2$, $z_2 = u_2 u_3$, $z_3 = u_3 u_4$, and $z_4 = u_4 u_5$. Omitting the case that is reduced to Type 6 , 9 , 12 , 13 , 14 , 15 , 16 , 17 , 19 , 20 , 21 , or 22, we obtain that 
$U \subset \langle z_1 u_2 , z_1 u_3 , z_3 u_4 , z_4 u_1 \rangle$ and so forth. □

Type 24: [11, 13, 24, 25]
We put $z_1 = u_1^2$, $z_2 = u_2 u_3$, $z_3 = u_3 u_4$, and $z_4 = u_4 u_5$. Omitting the case that is reduced to Type 9 , 10 , 11 , 12 , 13 , 14 , 15 , 19 , 20 , 21 , or 22, we obtain that 
$U \subset \langle z_1 u_2 , z_1 u_3 , z_3 u_4 , z_4 u_1 \rangle$ and that dim $U \leq 5$, or else $(U)^2 = \langle 0 \rangle$ and that dim $U \leq 6$. □

Type 25: [13, 14, 15, 22]
We put $z_1 = u_1 u_3$, $z_2 = u_1 u_4$, $z_3 = u_1 u_5$, and $z_4 = u_2^2$. Omitting the case that is reduced to Type 6 , 7 , 12 , 13 , 14 , 15 , 16 , 19 , 20 , 23 , or 24, we obtain that 
$U \subset \langle z_1 u_2 , z_1 u_3 , z_2 u_4 \rangle$ and so forth. □

Type 26: [12, 13, 14, 15]
We put $z_1 = u_1 u_2$, $z_2 = u_1 u_3$, $z_3 = u_1 u_4$, and $z_4 = u_1 u_5$. Omitting the case that is reduced to Type 7 , 14 , 16 , or 25, we obtain that 
$U \subset \langle z_1 u_2 , z_1 u_3 , z_1 u_4 , z_2 u_4 , z_3 u_4 , z_4 u_1 \rangle$ and that $z_1 u_2$, $z_1 u_4$, $z_1 u_4$, $z_2 u_4$, $z_3 u_4$, $z_5 u_5$ are linearly dependent, so dim $U \leq 5$, or else $(U)^2 = \langle 0 \rangle$ and that dim $U \leq 6$. □

Type 27: [12, 13, 14, 25]
We put $z_1 = u_1 u_2$, $z_2 = u_1 u_3$, $z_3 = u_1 u_4$, and $z_4 = u_2 u_5$. Omitting the case that is reduced to Type 7 , 9 , 13 , 14 , 16 , 17 , 21 , 22 , 23 , 24 , 25 , or 26, we obtain that 
$U \subset \langle z_1 u_3 , z_1 u_4 , z_1 u_5 , z_2 u_3 \rangle$ and so forth. □

Type 28: [12, 13, 24, 35]
We put $z_1 = u_1 u_2$, $z_2 = u_1 u_3$, $z_3 = u_2 u_4$, and $z_4 = u_2 u_5$. Omitting the case that is reduced to Type 9 , 13 , 16 , 17 , 21 , 22 , 23 , 24 , or 27, we obtain that 
$U \subset \langle z_1 u_3 , z_1 u_4 , z_2 u_3 \rangle$ and so forth. □

Type 29: [12, 14, 23, 35]
We put $z_1 = u_1 u_2$, $z_2 = u_1 u_4$, $z_3 = u_2 u_3$, and $z_4 = u_3 u_5$. Omitting the case that is reduced to Type 9 , 13 , 16 , 17 , 21 , 22 , 23 , 24 or 27, we get that 
$U \subset \langle z_1 u_2 , z_1 u_4 , z_2 u_3 \rangle$ and so on. □

Type 30: [12, 13, 23, 45]
We put $z_1 = u_1 u_2$, $z_2 = u_1 u_3$, $z_3 = u_2 u_3$, and $z_4 = u_4 u_5$. Omitting the case that is reduced to Type 2 , 15 , 16 , 21 , 22 , 24 , 27 , or 28, we obtain that 
$U \subset \langle z_1 u_2 , z_2 u_3 \rangle$ and so forth. □

Type 31: [11, 22, 34, 56]
We put $z_1 = u_1^2$, $z_2 = u_2^2$, $z_3 = u_3 u_4$, and $z_4 = u_3 u_4$. Omitting the case that is reduced to Type 8 , 10 , 12 , 18 , 19 , 20 , 21 , or 23, we obtain that 
$U \subset K z_1 u_2$ and so forth. □

Type 32: [11, 12, 34, 56]
We put $z_1 = u_1^2$, $z_2 = u_1 u_2$, $z_3 = u_2 u_4$, and $z_4 = u_3 u_4$. Omitting the case that is reduced to Type 9 , 10 , 11 , 13 , 19 , 21 , 22 , 23 , 24 , or 31, we obtain that 
$U \subset K z_1 u_2$ and so forth. □

Type 33: [11, 23, 24, 56]
We put $z_1 = u_1^2$, $z_2 = u_2 u_3$, $z_3 = u_3 u_4$, and $z_4 = u_3 u_4$. Omitting the case that is reduced to Type 6 , 12 , 13 , 14 , 15 , 19 , 20 , 21 , 22 , 23 , 24 , 25 , 27 , 29 , 30 , 31 , or 32, we obtain that 
$U \subset K z_2 u_4$ and so forth. □

Type 34: [12, 13, 14, 56]
We put $z_1 = u_1 u_2$, $z_2 = u_1 u_3$, $z_3 = u_1 u_4$, and $z_4 = u_3 u_5$. Omitting the case that is reduced to Type 7 , 14 , 16 , 21 , 22 , 24 , 25 , 26 , 27 , 29 , 30 , or 33, we obtain that 
$U \subset \langle z_1 u_3 , z_1 u_4 , z_4 u_1 \rangle$ and so forth. □

Type 35: [12, 13, 24, 56]
We put $z_1 = u_1 u_2$, $z_2 = u_1 u_3$, $z_3 = u_2 u_4$, and $z_4 = u_3 u_6$. Omitting the case that is reduced to Type 9 , 13 , 16 , 17 , 21 , 22 , 23 , 24 , 27 , 28 , 29 , 30 , 32 , 33 , or 34, we obtain that 
$U \subset \langle z_1 u_3 , z_4 u_1 \rangle$ and so forth. □

Type 36: [13, 14, 25, 26]
We put $z_1 = u_1 u_3$, $z_2 = u_1 u_4$, $z_3 = u_2 u_5$, and $z_4 = u_2 u_5$. Omitting the case that is reduced to Type 21 , 22 , 24 , 27 , 29 , 30 , 33 , 34 , or 35, we obtain that 
$U \subset \langle z_3 u_4 , z_4 u_1 \rangle$ and so on. □

Type 37: [11, 23, 45, 67]
We put $z_1 = u_1^2$, $z_2 = u_2 u_3$, $z_3 = u_4 u_5$, and $z_4 = u_6 u_7$. Omitting the case that is reduced to Type 19, 21, 23, 31, 32, 33, or 35, we obtain that $U^0 = \langle 0 \rangle$ and so forth. □

**Type 38: [12, 13, 45, 67]**

We put $z_1 = u_1 u_2$, $z_2 = u_3 u_4$, $z_3 = u_5 u_6$, and $z_4 = u_7 u_8$. Omitting the case that is reduced to Type 21, 22, 24, 27, 29, 30, 32, 33, 34, 35, 36, or 37, we obtain that $U^0 \subseteq K z_1 u_3$ and so on. □

**Type 39: [12, 34, 56, 78]**

We put $z_1 = u_1 u_2$, $z_2 = u_3 u_4$, $z_3 = u_5 u_6$, and $z_4 = u_7 u_8$. Omitting the case that is reduced to Type 32, 35, 37, or 38, we obtain that $U^0 = \langle 0 \rangle$ and so forth. □

**Corollary 4.** If $\dim U^0 = 4$ and $U^0 = Z$, then $\dim U Z \leq 5$ or else $Z^2 = \langle 0 \rangle$ and, in either case, $\dim U Z \leq 6$.

**Remark on Theorem 5.**

One cannot replace the number 6 in the theorem with any other less values. In order to show this, we shall construct the example of Bernstein algebra in which $\dim U^0 = 4$ and $U^0 = 6$ (and $(U^0)^2 = \langle 0 \rangle$).

**Example.**

Let $A = \langle e, u_1, \ldots, u_{10}, z_1, \ldots, z_6 \rangle$ be a commutative 15-dimensional algebra having the following multiplication table:

- $e^2 = e$, $eu = \frac{1}{2}u$, $ez = 0$, $u^2 = z$
  \[ (i = 1, \ldots, 10; j = 1, \ldots, 4) \]
- $u_j u_2 = \alpha z_1 + \frac{1}{4} \alpha^{-1} z_2$
- $u_j u_3 = 2 \alpha \beta z_1 + \frac{1}{8} (\alpha \beta)^{-1} z_2$
- $u_j u_4 = 4 \alpha \beta z_1 + \frac{1}{16} (\alpha \beta)^{-1} z_2$
- $u_j u_5 = \beta z_2 + \frac{1}{4} \beta^{-1} z_2$
- $u_j u_6 = 2 \beta z_2 + \frac{1}{8} (\beta)^{-1} z_2$
- $u_j u_7 = \gamma z_3 + \frac{1}{4} \gamma^{-1} z_4$
- $u_j u_8 = u_3 u_4 = u_6 u_7$
- $u_j u_9 = u_8 u_4 = u_9 u_4 = u_{10}$
- $u_j u_1 = 2 \alpha \beta u_4$
- $u_j u_1 = -2 \alpha \beta u_5$
- $u_j u_1 = -2 \alpha \beta u_6$
- $u_j u_1 = -2 \alpha \beta u_7$

$z_4 u_2 = -4 \beta \gamma u_9$, $z_4 u_3 = -2 \gamma u_10$,

where $\alpha, \beta, \gamma$ are arbitrary nonzero elements in $K$, and other products are zero. Then one can see that $A$ is a Bernstein algebra having the decomposition $A = Ke + U + Z$ with respect to the idempotent $e$ with $U = \langle u_1, \ldots, u_{10} \rangle$, $Z = \langle z_1, \ldots, z_6 \rangle$ and, moreover, that it satisfies $U^0 = Z$, $U^0 = \langle u_3, \ldots, u_{10} \rangle$ and $(U^0)^2 = \langle 0 \rangle$.

We hope to generalize the relation between $\dim U^0$ and $\dim U^0$ to the case of $\dim U^0 > 4$. For that purpose it may be more desirable to prove Theorem 3, Theorem 4, and Theorem 5 in rather conceptual method than such computational one as given here.

**References**

1) A. Wörz-Busekros, *Algebra in Genetics*  
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